酉辛群上的调和分析(I)
——Fourier级数的收敛判别法*

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§1.1　引　　言

华罗庚[2]龚昇[3]~[8]钟家庆[9]研究了酉群U_n及旋转群$SO(n)$上的调和分析的各种问题，取得了丰富的结果。我们沿用他们的方法，来讨论酉辛群$USp(2n)$上的调和分析，得出相应的结果。

$2n$阶酉辛群$USp(2n)$是适合

$$U\bar{U}'=I\text{，}UJU'=J \tag{1.1.1}$$

的$2n$阶酉辛方阵的集合，而

$$J=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\dotplus\begin{pmatrix}0&1\\-1&0\end{pmatrix}\dotplus\cdots\dotplus\begin{pmatrix}0&1\\-1&0\end{pmatrix}.\quad\text{（共}n\text{项）} \tag{1.1.2}$$

习知[10]，$USp(2n)$的单值既约表示由n个非负整数$f=(f_1,f_2,\cdots,f_n)$，$f_1\geqslant f_2\geqslant\cdots\geqslant f_n\geqslant 0$，表征，$f$称为标签。记$A_f(U)$为$U\in USp(2n)$的标签为$f$的酉表示，$N(f)$是$A_f(U)$的阶。若

$$A_f(U)=\left(a^f_{ij}(U)\right)_{1\leq i,j\leq N(f)}. \tag{1.1.3}$$

置

$$\varphi^f_{ij}(U)=\sqrt{\frac{N(f)}{C}}a^f_{ij}(U) \tag{1.1.4}$$

及

$$\Phi_f(U)=\left(\varphi^f_{ij}(U)\right)_{1\leq i,j\leq N(f)}, \tag{1.1.5}$$

式中C是$USp(2n)$的体积，它等于

$$\frac{2^{2n^2+\frac{n}{2}}\pi^{n^2+n}}{(2n-1)!\,(2n-3)!\cdots3!\cdot1!}. \tag{1.1.6}$$

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设$u(U)$是$USp(2n)$上可积函数，其Fourier级数为

$$\sum_{f\geq 0} tr(C_f \Phi'_f(U)) , \tag{1.1.7}$$

其中

$$C_f = \int_{USp(2n)} u(V)\overline{\Phi_f(V)}\dot{V} . \tag{1.1.8}$$

$u(U)$的Fourier级数(1.1.7)的部分和为

$$S_N(U) = \sum_{N\geq l_1 > e_2 > \cdots > l_n > 0} tr(C_f \Phi'_f(U)), \tag{1.1.9}$$

其中

$$l_1 = f_1 + n,\ l_2 = f_2 + n - 1, \cdots, l_n = f_n + 1,\ l = (l_1, l_2, \cdots, l_n).$$

本文定出了$USp(2n)$上Fourier级数部分和$S_N(U)$的Dirichlet核后，证明了：若$u(U)$在$USp(2n)$上具有n^2阶连续微商，则它的Fourier级数部分和$S_N(U)$收敛于$u(U)$，并且

$$|S_N(U) - u(U)| \leqslant A\left(\frac{\ln^{n^2} N}{N}\right)^{\frac{1}{n+1}} ,$$

其中A是绝对常数。本文最后给出了一个关于$u(U)$的Fourier级数绝对收敛判别法。

$USp(2n)$上调和分析的其它问题讨论将另文发表。

本文是在我们的导师华罗庚教授和龚昇教授的指导下完成的。

§1.2 Fourier级数部分和的Dirichlet核

由§1.1的定义，容易得到

$$S_N(U) = \frac{1}{C} \int_{USp(2n)} u(VU)\mathbf{D}_N(V)\dot{V} , \tag{1.2.1}$$

其中

$$\mathbf{D}_N(V) = \sum_{N\geq l_1 > l_2 > \cdots > l_n > 0} N(f) X_f(\dot{V}) \tag{1.2.2}$$

称为Fourier级数部分和(1.1.9)的Dirichlet核，而$X_f(\overline{V})$是$\overline{V}$的表示的特征。

以下证明：

定理1.2.1 $USp(2n)$可积函数$u(U)$的Fourier级数(1.1.7)的部分和(1.1.9)可表为(1.2.1)，而Dirichlet核(1.2.2)为

$$\frac{(-1)^n \det(d^{(2i-1)}(\varphi_j))_{1\leq i,j\leq n}}{(2n-1)!(2n-3)!\cdots 3!\cdot 1! \det(\sin(n-i+1)\varphi_j)_{1\leq i,j\leq n}} , \tag{1.2.3}$$

其中

$$d_N(\varphi) = \frac{\sin(N+\frac{1}{2})\varphi}{2\sin\frac{1}{2}\varphi}$$

是一个变数的 Dirichlet 核，而$e^{\pm i\varphi_1}, e^{\pm i\varphi_2}, \cdots, e^{\pm i\varphi_n}$是$\overline{V}$的特征根。

证. 由于$\overline{V}\in USp(2n)$，则有$V_1\in USp(2n)$[10]，使

$$\overline{V}=V_1\begin{pmatrix} e^{i\varphi_1} & & & & \\ & e^{-i\varphi_1} & & & \\ & & \ddots & & \\ & & & e^{i\varphi_n} & \\ & & & & e^{-i\varphi_n}\end{pmatrix}V_1^{-1},$$

其中 $e^{\pm i\varphi_1}, e^{\pm i\varphi_2}, \cdots, e^{\pm i\varphi_n}$是$\overline{V}$的特征根；又由于

$$X_f(\overline{V})=\frac{s(l)}{s(n)},$$

这里

$$S(t)=\begin{vmatrix} St_1(\varphi_1),\cdots,St_1(\varphi_n)\\ \cdots\cdots\cdots\cdots\cdots\cdots \\ St_n(\varphi_1),\cdots,St_n(\varphi_n)\end{vmatrix},$$

其中 $t=(t_1,t_2,\cdots,t_n), t_1>t_2>\cdots>t_n>0$，$S_q(\varphi)=2i\sin q\varphi$，而

$$N(f)=\lim_{\substack{\varphi_1\to 0\\ \cdots\\ \varphi_n\to 0}} X_f(\overline{V}),$$

它等于

$$\frac{(-1)^{\frac{n(n-1)}{2}}\begin{vmatrix} l_1, & l_2, & \cdots, & l_n\\ l_1^3, & l_1^3, & \cdots, & l_n^3\\ \cdots\cdots & & & \\ l_1^{2n-1}, & l_2^{2n-1}, & \cdots, & l_n^{2n-1}\end{vmatrix}}{(2n-1)!\,(2n-3)!\cdot\cdots\cdot 3!\cdot 1!},$$

所以，

$$\mathbf{D}_N(V)=\frac{(-1)^{\frac{n(n-1)}{2}}\sum\limits_{N\geq l_1>l_2>\cdots>l_n>0}\begin{vmatrix}\sin l_1\varphi_1,\cdots,\sin l_1\varphi_n\\ \cdots\cdots\cdots\cdots \\ \sin l_n\varphi_1,\cdots,\sin l_n\varphi_n\end{vmatrix}\begin{vmatrix} l_1, & \cdots, & l_n\\ l_1^3, & \cdots, & l_n^3\\ \cdots\cdots & & \\ l_1^{2n-1}, & \cdots, & l_n^{2n-1}\end{vmatrix}}{(2n-1)!\,(2n-3)!\cdot\cdots\cdot 3!\cdot 1!\det(\sin(n-i+1)\varphi_j)_{1\leq i,j\leq n}}$$

应用[8]中的引理，即得

$$\mathbf{D}_N(V)=\frac{(-1)^{\frac{n(n-1)}{2}}\det\left(\sum\limits_{l=1}^{N} l^{2i-1}\sin l\varphi_j\right)_{1\leq i,j\leq n}}{(2n-1)!\,(2n-3)!\cdot\cdots\cdot 3!\cdot 1!\det(\sin(n-i+1)\varphi_j)_{1\leq i,j\leq n}},$$

而

$$\sum_{l=1}^{N} l^{2i-1}\sin l\varphi_j=(-1)^i d_N^{(2i-1)}(\varphi_j),\quad i=1,2,\cdots,n.$$

即得(1.2.3)。

§1.3　Fourier级数的收敛判别法

形如

$$\Lambda=\begin{pmatrix} e^{i\varphi_1} & & & & \\ & e^{-i\varphi_1} & & & \\ & & \ddots & & \\ & & & e^{i\varphi_n} & \\ & & & & e^{-i\varphi_n} \end{pmatrix},\quad -\pi\leqslant\varphi_j\leqslant\pi,\ j=1,2,\cdots,n.$$

的酉辛矩阵的集合是$USp(2n)$的子群．$USp(2n)$关于Λ的傍系记作$[USp(2n)]$，那末，$USp(2n)$与$[USp(2n)]$的体积元素$USp(\dot{2}n)$与$[USp(\dot{2}n)]$之间有如下关系[10]

$$USp(\dot{2}n)=[USp(\dot{2}n)]\,2^{n^2+n}\prod_{j=1}^{n}\sin^2\varphi_j\prod_{1\leq i<j\leq n}(\cos\varphi_i-\cos\varphi_j)^2 d\varphi_1\cdots d\varphi_n. \tag{1.3.1}$$

若

$$\overline{V}=V_1\begin{pmatrix} e^{i\varphi_1} & & & & \\ & e^{-i\varphi_1} & & & \\ & & \ddots & & \\ & & & e^{i\varphi_n} & \\ & & & & e^{-i\varphi_n} \end{pmatrix}V_1^{-1},\ V,V_1\in USp(2n),$$

取$\Gamma\in USp(2n)$，使

$$\Gamma\begin{pmatrix} e^{i\varphi_1} & & & & \\ & e^{-i\varphi_1} & & & \\ & & \ddots & & \\ & & & e^{i\varphi_n} & \\ & & & & e^{-i\varphi_n} \end{pmatrix}\Gamma^{-1}=\begin{pmatrix} e^{i\varphi_{\nu_1}} & & & & \\ & e^{-i\varphi_{\nu_1}} & & & \\ & & \ddots & & \\ & & & e^{i\varphi_{\nu_n}} & \\ & & & & e^{-i\varphi_{\nu_n}} \end{pmatrix},$$

这里$(\nu_1,\nu_2,\cdots,\nu_n)$是$(1,2,\cdots,n)$的一个排列．再取$\Gamma_1\in USp(2n)$，使

$$\Gamma_1\begin{pmatrix} e^{i\varphi_{\nu_1}} & & & & & & & \\ & e^{-i\varphi_{\nu_1}} & & & & & & \\ & & \ddots & & & & & \\ & & & e^{i\varphi_{\nu_k}} & & & & \\ & & & & e^{-i\varphi_{\nu_k}} & & & \\ & & & & & \ddots & & \\ & & & & & & e^{i\varphi_{\nu_n}} & \\ & & & & & & & e^{-i\varphi_{\nu_n}} \end{pmatrix}\Gamma_1^{-1}=\begin{pmatrix} e^{i\varphi_{\nu_1}} & & & & & & & \\ & e^{-i\varphi_{\nu_1}} & & & & & & \\ & & \ddots & & & & & \\ & & & e^{i\varphi_{\nu_k}} & & & & \\ & & & & e^{-i\varphi_{\nu_k}} & & & \\ & & & & & \ddots & & \\ & & & & & & e^{i\varphi_{\nu_n}} & \\ & & & & & & & e^{-i\varphi_{\nu_n}} \end{pmatrix},$$

记

$$\overline{V}^*=V_1\Gamma_1\Gamma\begin{pmatrix} e^{i\varphi_1} & & & & \\ & e^{-i\varphi_1} & & & \\ & & \ddots & & \\ & & & e^{i\varphi_n} & \\ & & & & e^{-i\varphi_n} \end{pmatrix}\Gamma^{-1}\Gamma_1^{-1}V_1^{-1},$$

由群上积分的不变性[1]，有

$$\frac{1}{C}\int_{USp(2n)} u(V^*U)\mathbf{D}_N(V)\dot{V}=\frac{1}{C}\int_{USp(2n)} u(VU)\mathbf{D}_N(V)\dot{V}.$$

如同[5]那样，作所有可能的V^*，将所有的$u(V^*U)$加起来，除以$2^n n!$，得$u^*(VU)$，有

$$S_N(U)=\frac{1}{C}\int_{USp(2n)} u^*(VU)\mathbf{D}_N(V)\dot{V}.$$

置

$$g(\varphi_1,\varphi_2,\cdots,\varphi_n)=\frac{1}{C}\int_{[USp(2n)]} u^*(VU)[\dot{V}],\tag{1.3.2}$$

这样，$g(\varphi_1,\varphi_2,\cdots,\varphi_n)$ 是对称周期函数，而且是偶函数，并有

$$S_N(U)=2^{n^2+n}\int\cdots\int_{\pi\geq\varphi_1\geq\cdots\geq\varphi_n\geq-\pi} g(\varphi_1,\varphi_2,\cdots,\varphi_n)\mathbf{D}_N(V)\prod_{j=1}^{n}\sin^2\varphi_j\times$$

$$\prod_{1\leq i<j\leq n}(\cos\varphi_i-\cos\varphi_j)^2 d\varphi_1\cdots d\varphi_n.$$

注意到

$$\frac{1}{C}\int_{USp(2n)}\mathbf{D}_N(V)\dot{V}=1,$$

就有

$$S_N(U)-u(U)=\frac{(-1)^n 2^{n^2+n}}{(2n-1)!(2n-3)!\cdot\cdots\cdot 3!1!}\int_{-\pi}^{\pi}\cdots\int(g(\varphi_1,\varphi_2,\cdots,\varphi_n)-u(U))\times$$

$$\frac{\det(d_N^{(2i-1)}(\varphi_j))_{1\leq i,j\leq n}}{\det(\sin(n-i+1)\varphi_j)_{1\leq i,j\leq n}}\prod_{j=1}^{n}\sin^2\varphi_j\prod_{1\leq i<j\leq n}(\mathrm{con}\varphi_i-\mathrm{con}\varphi_j)^2 d\varphi_1\cdots d\varphi_n.\tag{1.3.3}$$

容易证明：

$$\prod_{j=1}^{n}\sin^2\varphi_j\prod_{1\leq i<j\leq n}(\cos\varphi_i-\cos\varphi_j)^2=2^{-n(n-1)}\begin{vmatrix}\sin n\varphi_1,\cdots,\sin n\varphi_n\\ \cdots\cdots\cdots\cdots\cdots\\ \sin\varphi_1,\ \cdots,\sin\varphi_n\end{vmatrix}^2.\tag{1.3.4}$$

于是，(1.3.3)变为

$$\frac{(-1)^n 2^{-n}}{(2n-1)!(2n-3)!\cdot\cdots\cdot 3!1!}\int_{-\pi}^{\pi}\cdots\int(g(\varphi_1,\varphi_2,\cdots,\varphi_n)-u(U))$$

$$\times\begin{vmatrix}\sin n\varphi_1,\cdots,\sin n\varphi_n\\ \cdots\cdots\cdots\cdots\cdots\\ \sin\varphi_1,\ \cdots,\ \sin\varphi_n\end{vmatrix} d_N^{1}(\varphi_1)d_N^{(3)}(\varphi_2)\cdots d_N^{(2n-1)}(\varphi_n)d\varphi_1\cdots d\varphi_n.\tag{1.3.5}$$

注意到$g(\varphi_1,\varphi_2,\cdots,\varphi_n)$的周期性，假定$u(U)\in C^{n^2}$，$C^k$表$USp(2n)$上具有$k$次连续微商的函数全体，进行分部积分，(1.3.3)变为

$$\frac{(-1)^n 2^{2n}}{(2n-1)!\,(2n-3)!\cdot\cdots\cdot 3!\,1!}\int_{-\pi}^{\pi}\cdots\int\left\{\frac{\partial}{\partial\varphi_1}\frac{\partial^3}{\partial\varphi_2^3}\cdots\frac{\partial^{2n-1}}{\partial\varphi_n^{2n-1}}(g(\varphi_1,\varphi_2,\cdots,\varphi_n)-u(U))\right.$$

$$\left.\times\begin{vmatrix}\sin n\varphi_1,\cdots,\sin n\varphi_n\\ \cdots\cdots\cdots\cdots\\ \sin\varphi_1,\ \cdots,\ \sin\varphi_n\end{vmatrix}\right\} d_N(\varphi_1)\cdots d_N(\varphi_n)d\varphi_1\cdots d\varphi_n. \tag{1.3.6}$$

以下来证明：

定理1.3.1 若$u(U)\in C^{n^2}$，那末，它的Fourier级数的部分和$S_N(U)$收敛于$u(U)$，并且有

$$|S_N(U)-u(U)|\leqslant A\left(\frac{\ln^{n^2}N}{N}\right)^{\frac{1}{n+1}},$$

其中A是绝对常数。

证。 如同[5]，置

$$I_{\mu_1\mu_2\cdots\mu_n}=\int_{-\pi}^{\pi}\cdots\int\frac{\partial^{\mu_1}}{\partial\varphi_1^{\mu_1}}\cdots\frac{\partial^{\mu_n}}{\partial\varphi^{\mu_n}}(g(\varphi_1,\varphi_2,\cdots,\varphi_n)-u(U))$$

$$\times\frac{\partial^{\nu_n}}{\partial\varphi_1^{\nu_n}}\cdots\frac{\partial^{\nu_n}}{\partial\varphi_1^{\nu_n}}\begin{vmatrix}\sin n\varphi_1,\cdots,\sin n\varphi_n\\ \cdots\cdots\cdots\cdots\\ \sin\varphi_1,\ \cdots,\ \sin\varphi_n\end{vmatrix} d_N(\varphi_1)\cdots d_N(\varphi_n)d\varphi_1\cdots d\varphi_n, \tag{1.3.7}$$

其中$\mu_k+\nu_k=2k+1$，$\mu_k\geqslant 0$，$\nu_k\geqslant 0$，$k=1,2,\cdots,n$。于是，

$$|S_N(U)-u(U)|\leqslant\frac{2^{2n}}{(2n-1)!\,(2n-3)!\cdot\cdots\cdot 3!\,1!}\sum_{(\mu_1,\cdots\mu_n)}|I_{\mu_1\cdots\mu_1}|.$$

先考虑$I_{0\cdots 0}$，它等于

$$\int_{-\pi}^{\pi}\cdots\int(g(\varphi_1,\cdots,\varphi_n)-u(U))\,\frac{\partial}{\partial\varphi_1}\frac{\partial^3}{\partial\varphi_1^3}\cdots\frac{\partial^{2n-1}}{\partial_n^{2n-1}}\begin{vmatrix}\sin n\varphi_1,\cdots,\sin n\varphi_n\\ \cdots\cdots\cdots\cdots\\ \sin\varphi_1,\ \cdots,\ \sin\varphi_n\end{vmatrix}$$

$$\times d_N(\varphi_1)\cdots d_N(\varphi_n)d\varphi_1\cdots d\varphi_n. \tag{1.3.8}$$

设$\pi\geqslant\varphi_{j_1}\geqslant\varphi_{j_2}\geqslant\cdots\geqslant\varphi_{j_n}\geqslant-\pi$，这里$(j_1,j_2,\cdots,j_n)$是$(1,2,\cdots,n)$的一个排列，我们只要考虑其中一个即可，例如，

$$\pi\geqslant\varphi_1\geqslant\varphi_2\geqslant\cdots\geqslant\varphi_n\geqslant-\pi.$$

将此积分区域分解为：

$$R_1:\ \delta\geqslant\varphi_1\geqslant\varphi_2\geqslant\cdots\cdots\cdots\geqslant\varphi_n\geqslant-\pi,$$

$$R_2:\ \pi\geqslant\varphi_1\geqslant\delta\geqslant\varphi_2\geqslant\cdots\cdots\geqslant\varphi_n\geqslant-\pi,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$R_n:\ \pi\geqslant\varphi_1\geqslant\varphi_2\geqslant\cdots\cdots\geqslant\delta\geqslant\varphi_n\geqslant-\pi$$

$$R_{n+1}:\ \pi\geqslant\varphi_1\geqslant\varphi_2\geqslant\cdots\cdots\cdots\geqslant\varphi_n\geqslant\delta,$$

这里取$0<\delta<1$记积分$I_{0\cdots 0}$，其积分区域在R_j上者为I_j，于是，

$$I_{0\cdots 0}=I_1+I_2+\cdots+I_{n+1}.$$

习知[11]，

$$\int_{\delta}^{\pi} f(t)d_N(t)dt = O\left(\frac{1}{\delta N}\right)$$

及

$$\int_{-\pi}^{\pi} |d_N(t)|\, dt = O(\ln N),$$

这里假定 $f(t)$ 在 (o,π) 有连续的导函数。因此，

$$I_2 = O\left(\frac{\ln^{n-1}N}{\delta N}\right).$$

同理有

$$I_p = O\left(\frac{\ln^{n-p+1}N}{\delta^{p-1}N}\right), \quad p = 2, 3, \cdots, n+1.$$

再将 R_1 分解为：

$$Q_1:\ \delta \geqslant \varphi_1 \geqslant \varphi_2 \geqslant \cdots\cdots\cdots \geqslant \varphi_n \geqslant -\delta,$$

$$Q_2:\ \delta \geqslant \varphi_1 \geqslant \varphi_2 \geqslant \cdots\cdots \geqslant \varphi_{n-1} \geqslant -\delta \geqslant \varphi_n \geqslant -\pi,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Q_{n+1}:\ \delta \geqslant \varphi_1 \geqslant -\delta \geqslant \varphi_2 \geqslant \cdots\cdots \geqslant \varphi_n \geqslant -\pi.$$

于是 I_1 分解为 J_p，它等于

$$\int\cdots\int_{Q_p} (g(\varphi_1, \varphi_2, \cdots, \varphi_n) - u(U)) \frac{\partial}{\partial \varphi_1} \frac{\partial^3}{\partial \varphi_2^3} \cdots \frac{\partial^{2n-1}}{\partial \varphi_n^{2n-1}} \begin{vmatrix} \sin n\varphi_1, \cdots, \sin n\varphi_n \\ \cdots\cdots\cdots\cdots \\ \sin\varphi_1, \ \cdots, \ \sin\varphi_n \end{vmatrix}$$

$$\times d_N(\varphi_1) \cdots d_N(\varphi_n)\, d\varphi_1 \cdots d\varphi_n,$$

$$p = 1, 2, \cdots, n+1.$$

同样可以证明：

$$J_p = O\left(\frac{\ln^{n-p+1}N}{\delta^{p-1}N}\right), \quad p = 2, 3, \cdots, n+1.$$

现在考虑 J_1。由于

$$\delta \geqslant \varphi_1 \geqslant \varphi_2 \geqslant \cdots \geqslant \varphi_n \geqslant -\delta,$$

则有

$$g(\varphi_1, \varphi_2, \cdots, \varphi_n) - u(U) = w(u, \delta),$$

这里 w 是 $u(U)$ 的连续模；又因为 $u(U) \in C^{n^2}$，所以，

$$J_1 = O(\delta \ln^n N).$$

于是得到

$$I_{0\cdots 0} = O(\delta \ln^n N) + O\left(\frac{\ln^{n-1}N}{\delta N}\right) + \cdots + O\left(\frac{\ln^{n-p+1}N}{\delta^{p-1}N}\right) + \cdots + O\left(\frac{1}{\delta^n N}\right).$$

取 $\delta = (N \ln^n N)^{\frac{-1}{n+1}}$，就有

$$I_{0\cdots 0} = O\left(\left(\frac{\ln^{n^2}N}{N}\right)^{\frac{1}{n+1}}\right).$$

再考虑$I_{\mu_1\cdots\mu_n}$，这里$\mu_1,\mu_2,\cdots\mu_n$不全为零。若$\mu_1,\mu_2,\cdots,\mu_n$中有奇数，这时，$\nu_1,\nu_2,\cdots,\nu_n$中有偶数，例如$\nu_j(1\leqslant j\leqslant n)$是偶数，则(1.3.7)等于

$$\int_{-\pi}^{\pi}\cdots\int\frac{\partial^{\mu_1}}{\partial\varphi_1^{\mu_1}}\cdots\frac{\partial^{\mu_n}}{\partial\varphi_n^{\mu_n}}(g(\varphi_1,\varphi_2,\cdots,\varphi_n)-u(U))\sin\varphi_j F(\varphi_1,\cdots,\varphi_n)d\varphi_1\cdots d\varphi_n,$$

其中$F(\varphi_1,\cdots,\varphi_n)$是可微函数。由此易知，

$$I_{\mu_1\mu_2\cdots\mu_n}=O\left(\frac{\ln^{n-1}N}{N}\right);$$

若$\mu_1,\mu_2\cdots,\mu_n$全为偶数，则$\nu_1,\nu_2,\cdots\nu_n$全为奇数，这时至少是二个ν_i，ν_j相等$(i\neq j)$，由此易得：

$$\frac{\partial^{\nu_1}}{\partial\varphi_1^{\nu_1}}\cdots\frac{\partial^{\nu_n}}{\partial\varphi_n^{\nu_n}}\begin{vmatrix}\sin n\varphi_1,\cdots,\sin n\varphi_n\\ \cdots\cdots\cdots\cdots\\ \sin\varphi_1,\cdots,\sin\varphi_n\end{vmatrix}=\sin\frac{\varphi_i-\varphi_j}{2}G(\varphi_1,\cdots,\varphi_n),$$

式中$G(\varphi_1,\cdots,\varphi_n)$是可微函数。从而

$$I_{\mu_1\cdots\mu_n}=O\left(\frac{\ln^{n-1}N}{N}\right).$$

综上所述，得到

$$|S_N(U)-u(U)|\leqslant A\left(\frac{\ln^{n^2}N}{N}\right)^{\frac{1}{n-1}},$$

其中A是绝对常数。

§1.4　Fourier 级数的绝对收敛

如同[5]，我们也可以讨论$USp(2n)$上Fourier级数的绝对收敛问题。

设$u(U)$是$USp(2n)$上可积函数，则有

$$C_f\overline{C}_f'=\iint u(V)u)U)\overline{\Phi_f(V)}\Phi_f'(U)\dot{V}\dot{U}$$

$$=\frac{N(f)}{C}\iint u(V)u(WV)A_f(W')\dot{V}\dot{W}.$$

从而，

$$tr(c_f\bar{c}_f')=\frac{N(f)}{C}\iint u(V)u(WV)X_f(W)\dot{W}\dot{V}.$$

置
$$\psi(W)=\int_{USp(2n)}u(V)u(WV)\dot{V},$$

得

$$\frac{tr(c_f\bar{c}_f')}{N(f)}=\frac{1}{C}\int_{USp(2n)}\psi(W)X_f(W)\dot{W}.$$

设$e^{i\varphi_1}$，$e^{-i\varphi_1},\cdots,e^{i\varphi_n}$，$e^{-i\varphi_n}$是$W$的特征根，置

$$h(\varphi_1,\varphi_2,\cdots,\varphi_n)=\frac{2^{3n}\pi^n}{C}\int_{[USp(2n)]}\psi(W)[\dot{W}],$$

于是有

$$\frac{tr(c_f\bar{c}_f')}{N(f)}=\frac{1}{(2\pi)^n}\int\cdots\int_{\pi\geq\varphi_1\geq\cdots\geq\varphi_n\ -\pi} h(\varphi_1,\varphi_2,\cdots,\varphi_n)\begin{vmatrix}\sin l_1\varphi_1,\cdots,\sin l_1\varphi_n\\ \cdots\cdots\cdots\cdots\\ \sin l_n\varphi_1,\cdots,\sin l_n\varphi_n\end{vmatrix}$$

$$\times\begin{vmatrix}\sin n\varphi_1,\cdots,\sin n\varphi_n\\ \cdots\cdots\cdots\cdots\\ \sin\varphi_1,\cdots,\sin\varphi_n\end{vmatrix}d\varphi_1\cdots d\varphi_n.$$

如同§1.3那样，用h^*代替h，h^*是对称偶数函数而且有

$$\frac{tr(c_f\bar{c}_f')}{N(f)}=\frac{1}{(2\pi)^n}\int\cdots\int_{\pi\geq\varphi_1\geq\cdots\geq\varphi_n\geq-\pi} h^*(\varphi_1,\cdots,\varphi_n)\begin{vmatrix}\sin l_1\varphi_1,\cdots,\sin l_1\varphi_n\\ \cdots\cdots\cdots\cdots\\ \sin l_n\varphi_1,\cdots,\sin l_n\varphi_n\end{vmatrix}$$

$$\times\begin{vmatrix}\sin n\varphi_1,\cdots,\sin n\varphi_n\\ \cdots\cdots\cdots\cdots\\ \sin\varphi_1,\cdots,\sin\varphi_n\end{vmatrix}d\varphi_1\cdots d\varphi_n.$$

$$=\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int h^*(\varphi_1,\cdots\varphi_n)\sin l_1\varphi_1\cdots\sin l_n\varphi_n\begin{vmatrix}\sin n\varphi_1,\cdots,\sin n\varphi_n\\ \cdots\cdots\cdots\cdots\\ \sin\varphi_1,\cdots,\sin\varphi_n\end{vmatrix}d\varphi_1\cdots d\varphi_n.$$

置

$$H(\varphi_1,\varphi_2,\cdots,\varphi_n)=h^*(\varphi_1,\cdots,\varphi_n)\begin{vmatrix}\sin n\varphi_1,\cdots,\sin n\varphi_n\\ \cdots\cdots\cdots\cdots\\ \sin\varphi_1,\cdots,\sin\varphi_n\end{vmatrix},\tag{1.4.1}$$

于是，

$$\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int H(\varphi_1,\cdots,\varphi_n)\sin l_{i_1}\varphi_1\cdots\sin l_{i_n}\varphi_n d\varphi_1\cdots d\varphi_n=\delta_{1,2\cdots n}^{i_1i_2\cdots i_n}\frac{tr(c_f\bar{c}_f')}{N(f)}.\tag{1.4.2}$$

由于$H(\varphi_1,\cdots,\varphi_n)$是奇函数，所以它的多重Fourier级数为

$$\sum_{l_1l_2\cdots l_n}d_{l_1l_2\cdots l_n}\sin l_1\varphi_1\cdots\sin l_n\varphi_n,\tag{1.4.3}$$

这里

$$d_{l_1l_2\cdots l_n}=\frac{tr(c_f\bar{c}_f')}{N(f)},\ 若\ l_1>l_2>\cdots>l_n>0;$$

$d_{l_1l_2\cdots l_n}=0$，若$l_1,l_2,\cdots,l_n$中至少有二个相同；

$d_{l_{i_1}l_{i_2}\cdots l_{i_n}}=\delta_{1,2\cdots n}^{i_1i_2\cdots i_n}d_{l_1l_2\cdots l_n}$，若$(i_1,i_2,\cdots i_n)$是$(1,2,\cdots,n)$的一个排列，且$l_1>l_2>\cdots>l_n>0$.

这样，(1.4.3)可写成

$$\sum_{l_1>l_2>\cdots>l_n>0}\frac{tr(c_f\bar{c}_f')}{N(f)}\begin{vmatrix}\sin l_1\varphi_1,\cdots,\sin l_1\varphi_n\\ \cdots\cdots\cdots\cdots\\ \sin l_n\varphi_1,\cdots,\sin l_n\varphi_n\end{vmatrix}. \tag{1.4.4}$$

记算子

$$\begin{vmatrix}\frac{\partial}{\partial\varphi_1},\cdots,\frac{\partial}{\partial\varphi_n}\\ \frac{\partial^3}{\partial\varphi_1^3},\cdots,\frac{\partial^3}{\partial\varphi_n^3}\\ \cdots\cdots\cdots\\ \frac{\partial^{2n-1}}{\partial\varphi_1^{2n-1}},\cdots,\frac{\partial^{2n-1}}{\partial\varphi_n^{2n-1}}\end{vmatrix} \tag{1.4.5}$$

为 $D^*\left(\frac{\partial}{\partial\varphi_1},\ \frac{\partial}{\partial\varphi_2},\ \cdots,\ \frac{\partial}{\partial\varphi_n}\right)$. 由于

$$\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int\left(D^*\left(\frac{\partial}{\partial\varphi_1},\cdots,\frac{\partial}{\partial\varphi_n}\right)H(\varphi_1,\cdots,\varphi_n)\right)\cos l_1\varphi_1\cdots\cos l_n\varphi_n d\varphi_1\cdots d\varphi_n$$

$$=\sum_{(i_1,\cdots,i_n)}\delta^{i_1i_2\cdots i_n}_{12\cdots n}\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int\left(\frac{\partial}{\partial\varphi_{i_1}}\ \frac{\partial^3}{\partial\varphi_{i_2}^3}\cdots\frac{\partial^{2n-1}}{\partial\varphi_i^{2n-1}}\ H(\varphi_1,\cdots,\varphi_n)\right)$$

$$\cdot\cos l_1\varphi_1\cdots\cos l_n\varphi_n d\varphi_1\cdots d\varphi_n$$

$$=\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int H(\varphi_1,\cdots,\varphi_n)\sin l_1\varphi_1\cdots\sin l_n\varphi_n d\varphi_1\cdots d\varphi_n\,(-1)^{\frac{n(n-1)}{2}}$$

$$\cdot\sum_{(i_1,\cdots,i_n)}\delta^{i_1,\cdots i_n}_{1,2\cdots n}l_{i_1}l_{i_2}^3\cdots l_{i_n}^{2n-1}$$

$$=(-1)^n(2n-1)!\,(2n-3)!\cdots3!\,1!\,tr(C_f\bar{C}_f'),$$

这里 $l_1>l_2>\cdots>l_n>0$，所以，$D^*\left(\frac{\partial}{\partial\varphi_1},\ \cdots,\ \frac{\partial}{\partial\varphi_n}\right)H(\varphi_1,\cdots,\varphi_n$的多重 *Fourier*级数为

$$(-1)^n(2n-1)!\,(2n-3)!\cdot\cdots3!\,1!\sum_{l_1>l_2>\cdots>l_n>0}tr(c_f\bar{c}_f)\begin{vmatrix}\cos l_1\varphi_1,\cdots,\cos l_1\varphi_n\\ \cdots\cdots\cdots\cdots\\ \cos l_n\varphi_1,\cdots,\cos l_n\varphi_n\end{vmatrix}. \tag{1.4.6}$$

同样，$\left(D^*\left(\frac{\partial}{\partial\varphi_1},\cdots,\frac{\partial}{\partial\varphi_n}\right)\right)^3H(\varphi_1,\cdots,\varphi_n)$的多重 Fourier 级数为

$$(-1)^n(2n-1)!(2n-3)!\cdot\cdots\cdot 3!1!)^3 \sum_{l_1>l_2>\cdots>l_n>0}(N(f))^2 tr(c_f\bar{c}_f)$$

$$\times\begin{vmatrix} \cos l_1\varphi_2,\cdots,\cos l_1\varphi_n \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cos l_n\varphi_1,\cdots,\cos l_n\varphi_n \end{vmatrix}; \qquad (1.4.7)$$

$\left(D^*\left(\frac{\partial}{\partial\varphi_1},\ \cdots,\ \frac{\partial}{\partial\varphi_n}\right)\right)^5 H(\varphi_1,\cdots,\varphi_n)$的多重 Fourier 级数为

$$(-1)^n((2n-1)!(2n-3)!\cdot\cdots\cdot 3!1!)^5 \sum_{l_1>l_2>\cdots>l_n>0}(N(f))^4 tr(c_f\bar{c}_f')$$

$$\times\begin{vmatrix} \cos l_1\varphi_1,\cdots,\cos l_1\varphi_n \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cos l_n\varphi_1,\cdots,\cos l_n\varphi_n \end{vmatrix}. \qquad (1.4.8)$$

如同[5]中那样，可以证明 $USp(2n)$上 Fourier 级数（1.1.7）的绝对收敛定理。

定理 1.4.1　若 $u(U)\in C^{n^2}$，且

$$D^*\left(\frac{\partial}{\partial\varphi_1},\ \cdots,\ \frac{\partial}{\partial\varphi_n}\right)H(\varphi_1,\cdots,\varphi_n)\in Lip(2,\alpha),$$

当　$\alpha>\frac{1}{r}-\frac{1}{2}$时，则级数

$$\sum_{f\geqslant 0}|tr(c_f,\bar{c}_f')|^r$$

收敛，但$1>r>\frac{2}{3}$.

定理 1.4.2　若　$u(U)\in C^{3n^2}$，且

$$\left(D^*\left(\frac{\partial}{\partial\varphi_1},\ \cdots,\ \frac{\partial}{\partial\varphi_n}\right)\right)^3 H(\varphi_1,\cdots,\varphi_n)\in Lip(2,\alpha),$$

那末，当 $\alpha>\frac{1}{r}-\frac{1}{2}$时，级数

$$\sum_{f\geqslant 0}\left(\sum_{j,k=0}^{N(f)}|C_{jk}^f||\varphi_f^{j_k}|\right)^{2r}$$

收敛，但$1>r>\frac{2}{3}$.

定理 1.4.3　若$u(U)\in C^{5n^2}$，且

$$\left(D^*\left(\frac{\partial}{\partial\varphi_1},\ \cdots,\ \frac{\partial}{\partial\varphi_n}\right)\right)^5 H(\varphi_1,\cdots,\varphi_n)\in Lip(2,\alpha),$$

则$u(U)$的 Fouriei 级数(1.1.7)绝对收敛，但 $\alpha>\frac{1}{2}$.

参 考 文 献

[1] 华罗庚，多复变函数论中的典型域的调和分析，科学出版社，(1959)；

[2] 华罗庚，紧致群上的连续函数所成空间中的一条收敛定理，科学记录新辑2：9(1958)，341—344；

[3] 龚 昇，酉群上的富里埃分析Ⅰ，数学学报，10：2(1960)，239—261；

[4] 龚 昇，酉群上的富里埃分析Ⅱ，数学学报，12：1(1962)，17—31；

[5] 龚 昇，酉群上的富里埃分析Ⅲ，数学学报，13：2(1963)，152—161；

[6] 龚 昇，酉群上的富里埃分析Ⅳ，数学学报，13：3(1963)，323—331；

[7] 龚 昇，酉群上的富里埃分析Ⅴ，数学学报，15：3(1965)，323—331；

[8] 龚 昇，旋转群上的富里埃级数的部分和，中国科学技术大学学报，9：1(1979)，25—30；

[9] 锺家庆，旋转群上的调和分析— Abel 求和，中国科学技术大学学报，9：1(1979)31—43；

[10] Weyl, H., The classical. group, (1946)；

[11] Zygmund, A., Trigonometric series. (1959).

Harmonic Analysis on Unitary—symplectic Group I. Convergence Criterion of Fourier Series

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Abstract

Profs. I. K. Hua[2], Kung Sun[3]-[8] and Zhung Jiaqin[9] have studied various problems about harmonic analysis on unitary group U_n and rotation group SO(n), and obtained abundent results. By the methods given by them, we study harmonic analysis on unitary-symplectic group USp(2n) and get similar results.

Let u(U) be an integrable function on USp(2n) and its Fourier series be

$$\sum_{f_1\geq f_2\geq\cdots\geq f_n\geq 0} tr(C_f\Phi_f'(U)) \tag{1.1}$$

where

$$\Phi_f(U)=\sqrt{\frac{N(f)}{C}}\,A_f(U),$$

$$C_f=\int_{USp(2n)}\mu(V)\Phi_f(\bar{V})\dot{V},$$

and let $A_f(U)$ be the unitary representation of USp(2n) with signature $f=(f_1,f_2,\cdots,f_n)$, where $f_1,f_2,\cdots,f_n$ are integers satisfying $f_1\geqslant f_2\geqslant\cdots\geqslant f_n\geqslant 0$, $N(f)=N(f_1,f_2,\cdots,f_n)$ be the order of the matrix $A_f(U)$, and C be volume of the unitary-symplectic group USp(2n), i. e.

$$C=\frac{2^{2n^2+\frac{n}{2}}\pi^{n^2+n}}{(2n-1)!(2n-3)!\cdots 3!1!}.$$

Then the partial sum of the Fourier series(1.1)is

$$S_N(U)=\sum_{N>l_1>l_2>\cdots>l_n>0} tr(C_f\Phi_f'(U)), \tag{1.2}$$

where $l_1=f_1+n,\cdots,l_n=f_n+1$.

In the present note, we proved

Theorem I The partial sum of the Fourier series (1.1) of the integrable function u(U) on USp(2n) can be expressed as

$$\frac{1}{C}\int_{USp(2n)}\mu(VU)\mathbf{D}_N(V)\dot{V}, \tag{1.3}$$

where the Dirichlet kernel

$$\mathbf{D}_N(V)=\frac{(-1)^n\det(d_N^{(2i-1)}(\varphi_j))_{1\leq i,\,j\leq n}}{(2n-1)!(2n-3)!\cdots 3!1!\det(\sin(n-i+1)\varphi_j)_{1\leq i,\,j\leq n}} \tag{1.4}$$

and

$$d_N(\varphi)=\frac{\sin\left(N+\frac{1}{2}\right)\varphi}{2\sin\frac{1}{2}\varphi}$$

is the Dirichlet kernel of one variable, $e^{\pm i\varphi_1},\cdots,e^{\pm i\varphi_n}$ are the characteristic roots of $\overline{V}$.

Theorem II If $u(U)\in C^{n^2}$, then the partial sum $S_N(U)$ of its Fourier series is convergent to u(U). Moreover,

$$|S_N(U)-u(U)|\leqslant A\left(\frac{\ln^{n^2}N}{N}\right)^{\frac{1}{n+1}},$$

where A is absolute constant.

Theorem III with $H(\varphi_1,\varphi_2,\cdots,\varphi_n)$ defined by(1.4.1), operator $D^*\left(\frac{\partial}{\partial\varphi_1},\cdots,\frac{\partial}{\partial\varphi_n}\right)$ defined by (1.4.5), in the present paper if $u(U)\in C^{n^2}$, and

$$\left(D^*\left(\frac{\partial}{\partial\varphi_1},\ \cdots,\ \frac{\partial}{\partial\varphi_n}\right)\right)^5 H(\varphi_1,\varphi_2,\cdots,\varphi_n)\in Lip(2,d),$$

then the Fourier series (1.1) of u(U) is **absolutely convergent** with $\alpha>\frac{1}{2}$.