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# BROWNIAN MOTION ON THE LINE*

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## Introduction

These lecture notes are based on a short course given in the fall of 1979. The audience consists of graduate students from several departments and some faculty auditors. They are supposed to know the elements of Brownian motion including the continuity of paths and the strong Markov property. as well as elementary martingale theory. The emphasis is on methodology and some inculcation is intended.

In this lecture $\{X(t),\ t\geq 0\}$ is a standard Brownian motion on $R=(-\infty,+\infty)$.

## §1. Exit and Return

Let $(a, b)$ be a finite interval and put

(1) $$\tau=\tau_{(a,b)}=\inf\{t>0: X(t)\notin(a,b)\}.$$

This is called the [first] exit time from $(a,b)$. If $x\notin[a,b]$, then clearly

(2) $$p^x\{\tau=0\}=1$$

by the continuity of paths. If $x=a$ or $x=b$, the matter is not so obvious because the inf in (1) is taken over $t>0$, not $t\geq 0$. Starting at $a$, it is conceivable that the path might move into $(a,b)$ without leaving it for some time. It is indeed true that (2) holds for $x=a$ and $x=b$ but the proof will be given later. Our immediate concern is: if the path starts at $x$, where $x\in(a,b)$, will it leave the interval eventually?

We have the crude inequality

(3) $$\sup_{x\in(a,b)} p^x\{\tau>1\}\leq\sup_{x\in(a,b)} p^x\{x(1)\in(a,b)\}.$$

* 1980年12月12日收到.

The number on the right side can be expressed in terms of a normal distribution, but it is sufficient to see that it is strictly less than one. Denote it by $\delta$. The next step is a basic argument using the Markovian character of the process. For any $x\in(a,b)$ and $n\geqslant 1$:

$$p^x\{\tau>n\}\leqslant E^x\{\tau>n-1;\ p^{x(n-1)}[\tau>1]\}\leqslant p^x\{\tau>n-1\}\cdot\delta. \tag{4}$$

In the above we have adopted the notation commonly used in Markov processes. It symbolizes the argument that if the path has not left $(a,b)$ at time $n-1$, then wherever it may be at that time, the probability does not exceed $\delta$ that it will remain in $(a,b)$ for another unit of time. It follows by induction on n that

$$p^x\{\tau>n\}\leqslant\delta^n \tag{5}$$

Since $\delta<1$ we obtain $p^x\{\tau=\infty\}=0$ by letting $n\to\infty$. This answers our question: the path will (almost surely) leave the interval eventually:

$$p^x\{\tau<\infty\}=1. \tag{6}$$

In fact the argument above yields more. For any $\varepsilon$ such that $0<\varepsilon<1/\delta$, we have

$$E^x\{e^{\varepsilon\tau}\}\leqslant\sum_{n=1}^{\infty}e^{\varepsilon n}p^x\{n-1<\tau\leqslant n\}\leqslant\sum_{n=1}^{\infty}e^{\varepsilon n}\delta^n<\infty. \tag{7}$$

In standard terminology this says that the random variable $\tau$ has a generating function $E^x\{e^{\theta\tau}\}$ which is finite for sufficiently small values of $\theta$. In particular it has finite moments of all orders.

At the exit time $\tau$, $x(\tau)=a$ or $x(\tau)=b$ by continuity. For any $x\in(-\infty,\infty)$, let us define

$$T_x=\inf\{t>0: x(t)=x\}. \tag{8}$$

This is called the *hitting time* of the set $\{x\}$. It is clear that

$$\tau_{(a,b)}=T_a\wedge T_b. \tag{9}$$

If we keep b fixed while letting $a\to-\infty$, then

$$\lim_{b\to\infty}\tau_{(a,b)}=T_b, \tag{10}$$

It is worthwhile to convince oneself that (10) is not only intuitively obvious but logically correct, for a plausible argument along the same line might mislead from (6) and (10) to:

$$p^x\{T_b<\infty\}=1. \tag{11}$$

The result is true but the argument above is fallacious[1] and its refutation is left as Exercise 1 below. We proceed to a correct proof.

**Proposition 1.** For every x and b in $(-\infty,\infty)$, (11) is true.

**Proof.** For simplicity of notation we suppose $x=0$ and $b=+1$. Let $a=-1$ in (7). Since the Brownian motion is symmetric with respect to the directions

1) 此点说穿了容易，但有误解可能，讲课时提出一问，果然有同事（应用数学家）落入罗网

right and left on the line, it should be clear that at the exit from $(-1, +1)$, the path will be at either endpoint with probability $1/2$. If it is at $+1$, then it hits $+1$. If not, the path is at $-1$, and we consider its further movement until the exit from $(-3, +1)$. According to the strong Markov property this portion of the path is like a new Brownian motion starting at $-1$, and is stochastically independent of the motion up to the hitting time of $-1$. Thus we may apply (7) again with $x=-1$, $a=-3$ and $b=+1$. Observing that $-1$ is the midpoint of $(-3,+1)$, we see that at the exit from the latter interval, the path will be at either endpoint with probability $1/2$. If it is at $+1$, then it hits $+1$. If not, the path is at $-3$; and we consider its further movement until the exit from $(-7, +1)$, and so on. After n such steps, if the path has not yet hit $+1$, then it will be at $-(2^n-1)$ and the next interval for exit to be considered is $(-(2^{n+1}-1), +1)$. These successive trials (attempts to hit $+1$) are independent, hence the probability that after n trials the path has not hit $+1$ is equal to $(\frac{1}{2})^n$. Therefore the probability is equal to $\lim_{n\to\infty}(\frac{1}{2})^n=0$ that the path will never hit $+1$; in other words the probability is 1 that it will hit $+1$ eventually.

The scheme described above is exactly the celebrated gambling strategy called "doubling the ante." [The origin of the name "martingale".] The gambler who is betting on the outcome of tossing a fair coin begins by staking $1 on the first outcome. If he wins he gets $1 and quits. If he loses he flips the coin again but doubles the stake to $2. If he wins then his net gain is $1 and he quits. If he loses the second time he has lost a total of $3. He then repeats the game but redoubles the stake to $4, and so on. The mathematical theory above shows that if the game is played in this manner indefinitely, the gambler stands to gain $1 sooner or later. Thus it is a "sure win" system, the only drawback being that one needs an infinite amount of money to play the system. To get a true feeling for the situation one should test the gambit by going to a roulette table[2] and bet repeatedly on "red or black", which is the nearest thing to a fair game in a casino.

There are many other proofs of Proposition 1, of which one will be given soon in §2. The proof above is however the most satisfying one because it reduces the problem to an old legend: if a coin is tossed indefinitely, sooner or later it will be heads!

**Exercise 1.** What can one conclude by letting $a\to-\infty$ in (6) while keeping x and b fixed? Figure out a simple Markov process (not Brownian motion) for which (6) is true but (11) is false.

**Exercise 2.** Show that (11) remains true for $x=b$, a moot point in the proof given above.

2) 据传大数学家 Poincaré 常去赌场观察记录红黑样本数列.

## §2. Time and Place

The results in §1 are fundamental but qualitative. We now proceed to obtain quantitative information on the time $\tau$ and the place $x(\tau)$ of the exit from $(a,b)$. Let us define

$$p_a(x)=p^x\{x(\tau)=a\}=p^x\{T_a<T_b\};\tag{1}$$
$$p_b(x)=p^x\{x(\tau)=b\}=p^x\{T_b<T_a\}.$$

It is a consequence of (1.7) that

$$p_a(x)+p_b(x)=1, \qquad \forall x\in(a,b).\tag{2}$$

In order to solve for $p_a(x)$ and $p_b(x)$ we need another equation. The neatest way is to observe that the Brownian motion process is a martingale. More pedantically, let $F_t$ be the $\sigma$-field generated by $\{x_s,\ 0\leqslant s\leqslant t\}$, then for each $x\in R$:

$$\{x_t,\ F_t,\ p^x\}\qquad \text{is a martingale.}\tag{3}$$

We leave the verification to the reader. The fundamental martingale stopping theorem then asserts that

$$\{x(t\wedge\tau),\ F(t\wedge\tau),\ p^x\}\qquad \text{is a martingale.}\tag{4}$$

The defining property of a martingale now yields

$$E^x\{x(0)\}=E^x\{x(t\wedge\tau)\}.\tag{5}$$

The left member above is equal to x; the right member is an "incalculable" quantity. Fortunately we can easily calculate its limit as $t\to\infty$. For almost every sample point $\omega$, $\tau(\omega)<\infty$ by (1.6) and $t\wedge\tau(\omega)=\tau(\omega)$ for $t\geqslant\tau(\omega)$, hence $\lim\limits_{t\to\infty}x(t\wedge\tau)=x(\tau)$ without even the continuity of $x(\cdot)$. Since

$$\sup_{0\leqslant t<\infty}|x(t\wedge\tau)|\leqslant|a|\vee|b|,\tag{6}$$

the dominated convergence theorem allows us to take the limit as $t\to\infty$ under the $E^x$ in (5) to conclude

$$x=E^x\{x(\tau)\}.\tag{7}$$

The novice must be warned that the verification of domination is absolutely *de rigueur* in such limit-taking, neglect of which has littered the field with published and unpublished garbage. [On this particular occasion the domination is of course trivial, but what if $\tau$ is replaced by $T_a$ for instance?]

Since $x(\tau)$ takes only the two values $a$ and $b$, (7) becomes

$$x=ap_a(x)+bp_b(x).\tag{8}$$

Solving (2) and (8) we obtain

$$p_a(x)=\frac{b-x}{b-a},\qquad p_b(x)=\frac{x-a}{b-a},\qquad x\in(a,\ b).\tag{9}$$

Note that (9) is valid only for $x \in [a, b]$, and implies $p_a(a) = 1$.

To obtain $E^x\{\tau\}$ we use another important martingale associated with the process:

(10) $\{x(t)^2 - t, F_t, p^x\}$ is a martingale.

Application of the stopping theorem gives

$$x^2 = E^x\{x(\tau \wedge t)^2 - (\tau \wedge t)\}.$$

Since $x(\tau \wedge t)^2 \leqslant a^2 + b^2$ and $E^x\{\tau\} < \infty$ (see §1) we can let $t \to \infty$ and use dominated convergence (how? ) to deduce

(11) $$x^2 = E^x\{x(\tau)^2 - \tau\} = a^2 p_a(x) + b^2 p_b(x) - E^x\{\tau\}.$$

Together with (9) this yields

(12) $$E^x\{\tau_{(a, b)}\} = (x - a)\ (b - x),$$

for all $x \in [a, b]$. For $x = a$ we have $E^a\{\tau_{(a, b)}\} = 0$, so $p^a\{\tau_{(a, b)} = 0\} = 1$.

From the last result it is tempting but fallacious to conclude that $p^a\{T_a = 0\} = 1$. *In the next exercise we give a correct proof of this fact using the symmetry of* Brownian motion, leaving the details to the reader.

**Exercise 3.** Prove the fundamental result

(13) $$p^x\{T_x = 0\} = 1, \qquad \textit{for all } x \in R.$$

We may take $x = 0$. Define $T^- = \inf\{t > 0 : x(t) < 0\}$ and $T^+ = \inf\{t > 0 : x(t) > 0\}$. From the results above deduce $p^0\{T^- = 0\} = 1$ and $p^0\{T^+ = 0\} = 1$; hence $p^0\{T_0 = 0\} = 1$.

**Exercise 4.** Show that starting at any x, the Borwnian path immediately crosses the x-level infinitely many times.

**Exercise 5.** Show that for any $x \neq b$ we have $E^x\{T_b\} = +\infty$.

**Exercise 6.** Let $x_n \to x$, then $p^x\{T_{x_n} \to 0\} = 1$.

**Exercise 7.** Let $\tau = \tau_{(a, b)}$ and $t > 0$. Show that $\varphi$ is concave in $[a,b]$ and hence continuous. Let $x_1, x_2, x \in [a, b]$ with $x = (1 - \lambda)x_1 + \lambda x_2$. Let $\tau' = \tau_{(x_1, x_2)}$. Check that $p_x(x_1) = 1 - \lambda$ and $p_x(x_2) = \lambda$, so it follows *from the strong Markov property* that

$$\varphi(y) \geqslant (1 - \lambda)\varphi(x_1) + \lambda\varphi(x_2).$$

**Exercise 8.** Use Exercise 6 and symmetry to show that for each t the maximum of $p^x\{\tau > t\}$ occurs at $x = (a + b)/2$.

## §3. A General Method

To obtain comprehensive information regarding the joint distribution of the time and place of exit from $(a, b)$, we introduce a more powerful martingale.

**Proposition 2.** For any real value of the parameter $\alpha$,

(1) $$\{\exp(\alpha x(t) - \frac{\alpha^2 t}{2}),\ F_t,\ p^X\} \quad \text{is a martingale.}$$

**Proof.** We begin with the formula

(2) $$E^X\{\exp(\alpha x(t))\} = \exp(\alpha x + \frac{\alpha^2 t}{2}).$$

The left member above being the probabilistic manifestation of the analytic formula

$$\int_{-\infty}^{\infty} e^{\alpha y} \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(x-y)^2}{2t}\}\, dy$$

its calculation is an exercise in beginner's calculus. When $\alpha = i\theta$ where $i = \sqrt{-1}$ and $\theta$ is real, (2) should be recognized as the familiar Fourier transform or characteristic function of the normal distribution commonly denoted by $N(x,t)$. It follows that if $0 \leqslant s < t$ then

(3) $$E^X\{\exp(\alpha x(t) - \frac{\alpha^2 t}{2}) \mid F_S\}$$

$$= \exp(\alpha x(s) - \frac{\alpha^2 t}{2}) E^X\{\exp(\alpha x(t) - x(s)) \mid F_S\}.$$

To be formal about it this time, let us put

$$\tilde{X}(u) = X(s+u) = X(s), \qquad for \quad u \geqslant 0;$$

Since the Brownian motion is a process with stationary independent increments, the "shifted" process $\tilde{x}$ is a standard Brownian motion which is independent of $F_s$. Using (2) we have

$$E^X\{\exp(\alpha(X(t) - X(s)) \mid F_s\} = E^0\{\exp(\alpha X(t-s))\} = \exp\left(\frac{\alpha^2(t-s)}{2}\right),$$

which confirms the assertion in (1).

For a few moments let us denote the martingale in (1) by $M(t)$. Then for every $x \in (a, b)$:

(4) $$e^{\alpha X} = E^X\{M(0)\} = E^X\{M(t \wedge \tau)\},$$

where $\tau = \tau_{(a,b)}$. Since

$$|M(t \wedge \tau)| \leqslant \exp(|\alpha|(|a| \wedge |b|)).$$

we obtain by dominated convergence just as in §2 that

(5) $$e^{\alpha X}=E^{X}\{M(\tau)\}=E^{X}\{\exp(\alpha a-\frac{\alpha^{2}\tau}{2});X(\tau)=a\}+E^{X}\{\exp(\alpha b-\frac{\alpha^{2}\tau}{2});\ X(\tau)=b\}.$$

Putting

(6) $$f_{a}(x)=E^{X}\{\exp(-\frac{\alpha^{2}\tau}{2});\ X(\tau)=a\},$$

$$f_{b}(x)=E^{X}\{\exp(-\frac{\alpha^{2}\tau}{2});X(\tau)=b\},$$

we have the equation

(7) $$e^{\alpha X}=e^{\alpha a}f_{a}(x)+e^{\alpha b}f_{b}(x),\qquad X\in(a,b).$$

We have also the equation

(8) $$f_{a}(x)+f_{b}(x)=E^{X}\{\exp(-\frac{\alpha^{2}\tau}{2})\}.$$

Unlike the situation in §2, these two equations do not yield the three unknowns involved. There are several ways of circumventing the difficulty. One is to uncover a third hidden equation—the reader should try to do so before peeking at the solution given below[3]. But this quickie method depends on a lucky quirk. By contrast, the method developed here, though much longer, belongs to the mainstream of probabilistic analysis and is of wide applicability. It is especially charming in the setting of $R^1$.

We begin with the observation that if x is the midpoint of $(a, b)$ then $f_a(x)=f_b(x)$ by symmetry so that in this case (7) is solvable for $f_a(x)$. Changing the notation we fix x and consider the interval $(x-h,\ x+h)$. We obtain from (7)

$$f_{X-h}(x)=\frac{e^{\alpha X}}{e^{\alpha(X-h)}+e^{\alpha(X+h)}}=\frac{1}{e^{\alpha h}+e^{-\alpha h}}$$

and consequently by (8)

(9) $$E^{X}\{\exp(-\frac{\alpha^{2}}{2}\cdot\tau_{(X-h,X+h)})\}=\frac{1}{ch(\alpha h)}.$$

Here *ch* denotes the "hyperbolic cosine" function, as *sh* denotes the "hyperbolic sine" function:

(10) $$ch\theta=\frac{e^{\theta}+e^{-\theta}}{2},\qquad sh\ \theta=\frac{e^{\theta}-e^{-\theta}}{2}.$$

With this foot in the door, we will push on to calculate $f_a(x)$.

3) 讲时问诸大众，无人看出妙计.

Recall $x \in (a,b)$, hence for sufficiently small $h>0$ we have $[x-h,x+h] \subset (a,b)$ and so

(11) $$\tau_{(x-h,x+h)} < \tau_{(a,b)}.$$

We shall denote $\tau_{(x-h,x+h)}$ by $\tau(h)$ below; observe that it can also be defined as follows:

(12) $$\tau(h) = \inf\{t>0: |X(t)-X(0)| \geq h\},$$

namely the first time that the path has moved a distance $\geq h$ (from whichever initial position). Now starting at x, the path upon its exit from $(x-h,x+h)$ will be at $x-h$ or $x+h$ with probability 1/2 each. From the instant $\tau(h)$ onward, the path moves as if it started at these two new positions by the strong Markov property. This verbal description is made symbolic below:

(13) $$E^x\{\exp(-\frac{a^2\tau}{2});\ X(\tau)=a\}$$

$$=E^x\{\exp(-\frac{a^2\tau(h)}{2});\ E^{X(\tau(h))}[\exp(-\frac{a^2\tau}{2});\ X(\tau)=a]\}.$$

It is crucial to understand why after the random shift of time given by $\tau(h)$, the "function to be integrated": $[\exp(-\frac{a^2\tau}{2});\ X(\tau)=a]$ does not change. This point is generally explained away by a tricky symbolism, but one should first perceive the truth with the naked eye. Anyway (13) may be written as

(14) $$f_a(x)=E^x\{\exp(-\frac{a^2\tau(h)}{2})\}\frac{1}{2}\{f_a(x-h)+f_a(x+h)\}.$$

Using (9) we may rewrite this as follows:

(15) $$\frac{f_a(x+h)-2f_a(x)+f_a(x-h)}{h^2}=\frac{2ch(ah)-2}{h^2}f_a(x).$$

Letting $h\downarrow 0$ we see that the left member in (15) converges to $a^2f_a(x)$. It is also immediate from (14) that

(16) $$f_a(x)<\frac{1}{2}\{f_a(x-h)+f_a(x+h)\},$$

valid for $a<x-h<x+h<b$. Since $f_a$ is also bounded, (16) implies $f_a$ is continuous in $(a,b)$, in fact convex (see e. g. [Courant, Differential and Integral Calculus Vol. II, p. 326]). Now if $f_a$ has a second derivative $f_a''$ in $(a,b)$, then an easy exercise in calculus shows that the limit as $h\downarrow 0$ of the left member in (15) is equal to $f_a''(x)$. What is less easy is to show that a close converse is also true. This is known as Schwarz's theorem on generalized second derivative, a basic lemma in Fourier series. We state it in the form needed below.

**Schwarz's Theorem.** Let $f$ be continuous in $(a,b)$ and suppose that

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}=\varphi(x) \quad \forall x\in(a,b)$$

where $\varphi$ is continuous in $(a,b)$. Then $f$ is twice differentiable and $f''=\varphi$ in $(a,b)$.

It is sufficient to prove this theorem when $\varphi\equiv 0$ in $(a,b)$, and a proof can be found in Titchmarsh, Theory of Functions, 2nd ed, p.431.[4] Since $f_a$ has been shown to be continuous, Schwarz's theorem applied to (15) yields the differential equation:

(17) $$f_a''(x)=\alpha^2 f(x), \quad x\in(a,b).$$

The most general solution of this equation is given by

(18) $$f_a(x)=Ae^{\alpha x}+Be^{-\alpha x},$$

where $A$ and $B$ are two arbitrary constants. To determine these we compute the limits of $f_a(x)$ as $x\to a$ and $x\to b$, from inside $(a,b)$. From (6) and (2.9) we infer that

(19) $$\overline{\lim_{x\to b}}\, f_a(x)\leqslant\lim_{x\to b} E^x\{X(\tau)=a\}=\lim_{x\to b}\frac{b-x}{b-a}=0;$$

$$\overline{\lim_{x\to b}}\, f_b(x)\leqslant\lim_{x\to b} E^x\{X(\tau)=b\}=\lim_{x\to b}\frac{x-a}{b-a}\leqslant 1.$$

Since $f_a\geqslant 0$, the first relation above shows that $\lim\limits_{x\to b} f_a(x)=0$, Using (7) we see that

$$e^{\alpha b}\leqslant e^{\alpha b}\underline{\lim_{x\to b}}\, f_b(x)$$

so $\lim\limits_{x\to b} f_b(x)=1$. Similarly we have

(20) $$\lim_{x\to a} f_a(x)=1, \qquad \lim_{x\to a} f_b(x)=0.$$

Thus we obtain from (18):

$$0=Ae^{\alpha b}+Be^{-\alpha b}, \qquad 1=Ae^{\alpha a}+Be^{-\alpha a}.$$

Solving for $A$ and $B$ and substituting into (18), we obtain

(21) $$f_a(x)=\frac{sh\,\alpha(b-x)}{sh\,\alpha(b-a)}, \qquad f_b(x)=\frac{sh\,\alpha(x-a)}{sh\,\alpha(b-a)},$$

where the second formula in (21) may be obtained from the first by interchanging a and $b$. Finally we have by (8):

(22) $$E^x\{\exp(-\frac{\alpha^2\tau}{2})\}=\frac{sh\,\alpha(b-x)+sh\,\alpha(x-a)}{sh\,\alpha(b-a)}.$$

**Exercise 9.** The quick way to obtain (21) is to use (7) for $-\alpha$ as well as $+\alpha$.

**Exercise 10.** Derive (2.9) from (21), and compute

$$E^x\{T_a;\ T_a<T_b\}.$$

4) W. Rogosinski, Fourier Series 小书中证法尤美.

Answer: $(b-x)(x-a)(2b-a-x)/3(b-a)$.

**Exercise 11.** Show that for $0\leqslant\theta<\pi^2/2(b-a)^2$, we have

$$E^X\{e^{\theta\tau(a,b)}\}=\frac{\cos(\sqrt{2\theta}(x-\frac{a+b}{2}))}{\cos((\sqrt{2\theta}(\frac{b-a}{2}))}.$$

Prove that $E^X\{e^{\theta\tau(a,b)}\}=+\infty$ for $\theta=\pi^2/2(b-a)^2$.

A third way to derive (21) will now be shown. Since (7) is valid for $-\alpha$ as well as $\alpha$, letting $a\to-\infty$ we obtain

$$e^{|\alpha|X}=e^{|\alpha|b}E^X\{\exp(-\frac{\alpha^2}{2}T_b)\}.$$

Changing $\alpha^2/2$ into $\lambda$, writing y for b and observing that the result is valid for any $x\neq y$ on account of symmetry of the Brownian motion with respect to right and left:

(23) $$E^X\{e^{-\lambda T_Y}\}=\exp(-\sqrt{2\lambda}|x-y|).$$

This equation holds also when $x=y$, but the argument above does not include this case. The little sticking point returns to haunt us! We can dispose of it as follows. If $y\to x$ then $T_Y\to T_X$ almost surely (proof?), hence the Laplace transform of $T_Y$ converges to that of $T_X$, and (23) takes on the limiting form

$$E^X\{e^{-\lambda T_X}\}=1.$$

It follows that $P_X\{T_X=0\}=1$, namely Exercise 3 again.

Recalling that $\tau_{(a,b)}=T_a\wedge T_b$, we can write down the following relations:

(24)
$$\begin{aligned}E^X\{e^{-\lambda T_b}\}&=E^X\{\exp(-\lambda(T_a\wedge T_b));\ T_b<T_a\}\\&\quad+E^X\{\exp(-\lambda\{T_a\wedge T_b));\ T_a<T_b\}\cdot E^a\{e^{-\lambda T_b}\}\\E^X\{e^{-\lambda T_a}\}&=E^X\{\exp(\lambda(T_a\wedge T_b));\ T_a<T_b\}\\&\quad+E^X\{\exp(-\lambda(T_a\wedge T_b);\ T_b<T_a\}\cdot E^b\{e^{-\lambda T_a}\}.\end{aligned}$$

Using (23) we see that these two equations can be solved for the two unknowns which are just $f_a(x)$ and $f_b(x)$ after a change of notation.