On Nonexistence of Some Type of One-to-one Mapping on R×R to R*

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In p. 32 of [1] the following problem is posed:

A problem on Peano mapping

Let R be the set of positive rational integers with usual operation $a+b\equiv s(a,b)$ and $a\cdot b\equiv m(a,b)$. Every one-to-one (Peano) mapping $c=p(a,b)$ on $R\times R$ to all R may serve so associate with $s(a,b)$ and $m(a,b)$ two functions σ and μ on R to R by the definitions $\sigma(c)=\sigma(p(a,b))=s(a,b)$, and $\mu(c)=\mu(p(a,b))=m(a,b)$. Does there exist a Peano mapping $P(a,b)$ such that "addition commutes with multiplication" in the sense that $\sigma(\mu(c))=\mu(\sigma(c))$ for all c of R? To illustrate, we note that the well-known Peano mapping $e=p(a,b)\equiv 2^{a-1}(2b-1)$ fails. For, $\sigma(\mu(14))=\sigma(\mu(2^{2-1}\cdot[2\cdot 4-1]))=\sigma(8)=\sigma(2^{4-1}\cdot[2\cdot 1-1])=5$, while $\mu(\sigma(14))=\mu(6)=\mu(2^{2-1}\cdot[2\cdot 2-1])=4$.

A whole answer is given by following

Theorem. There does not exist any one-to-one mapping with the above-mentioned property.

Proof. For each Peano mapping

$$p:\ R\times R\to R,$$

the p^{-1} exists since p is a one-to-one mapping.

$$p^{-1}:\ R\to R\times R.$$

We write

$$p^{-1}(c)=(a,b)=(x_c,y_c),\ x_c,y_c\in R.$$

Under this sign the condition

$$\sigma(\mu(c))=\mu(\sigma(c))$$

becames

$$\sigma(a\cdot b)=\mu(a+b),$$

or

$$\sigma(x_c\cdot y_c)=\mu(x_c+y_c). \tag{1}$$

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This equation holds for each $(x_c,y_c)\in R\times R$, because for each pair (x_c,y_c) there exists some c such that $p^{-1}(c)=(x_c,y_c)$. In particular, it may follow from this equation that

$$\sigma(7)=\sigma(1\times 7)=\mu(1+7)=\mu(2+6)=\sigma(2\times 6)=\sigma(3\times 4)=\mu(3+4)=\mu(7),$$

Put

$$p^{-1}(7)=(x_c,y_c).$$

Then

$$\sigma(7)=\mu(7)$$

means that

$$x_7+y_7=x_7y_7.$$

Note that

$$x_7,\ y_7\in R,$$

and

$$x_7\,|\,y_7,\ y_7\,|\,x_7.$$

We see that

$$x_7=y_7.$$

It follows that

$$2x_7=x_7^{\,2},$$

thus

$$x_7=y_7=2 \text{ (here 0 is omitted).} \tag{2}$$

Again it may follow from equation (1) that

$$\sigma(10)=\sigma(1\times 10^*)=\mu(1+10)=\mu(3+8)=\sigma(3\times 8)=\sigma(4\times 6)=\mu(4+6)=\mu(10).$$

It follows that

$$x_{10}=y_{10}=2 \tag{3}$$

by using similar method, where

$$(x_{10},y_{10})=p^{-1}(10).$$

(2) and (3) give us

$$p^{-1}(7)=(2,2)=p^{-1}(10),$$

or

$$7=p(2,2)=10.$$

This contradicts with one-to-one property of Peano mapping. Therefore each mapping with condition

$$\sigma(\mu(c))=\mu(\sigma(c)) \qquad (\forall c\in R)$$

can never be a Peano one. The theorem is proved.

Reference

[1] Ulam, S. M., A Collection of Mathematical Problems, 1960.