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用 Hermite 插值同时逼近函数及其导数*

郁定国

（杭州大学）

空间 $C^k_{[-1,1]}$ 系指在 $[-1, 1]$ 上具有 k 阶连续导数的实值函数全体．$f\in C^k_{[-1,1]}$则定义其范数$\|f\|_k$为：

$$\|f\|_k=\max_{0\leqslant\mu\leqslant k}\left(\sup_{x\in[-1,1]}|f^{\mu}(x)|\right).$$

对于节点组 $X_n=\{x_{1n}, x_{2n},\cdots,x_{nn}\}\subseteq[-1, 1]$, $n=1,2,\cdots$, 我们考虑 Hermite 插值算子 H_{2n-1}: $C^1_{[-1,1]}\to C^1_{[-1,1]}$．周知，并非对任何 $f\in C^1_{[-1,1]}$都有

$$\lim_{n\to\infty}\|f-H_{2n-1}(f)\|_1=0 \tag{1}$$

成立．（参见[1],[2]．）

P. Pottinger[3]对节点组X_n由第一类 Чебышев 多项式 $T_n(x)=\cos n\theta(x=\cos\theta)$ 的零点 $x_k=\cos\frac{2k-1}{2n}\pi(k=1,2,\cdots,n)$ 组成时，求出能使(1)成立的函数类．得有：

定理P (a) 若 $f\in C^2_{[-1,1]}$, 则 $\lim_{n\to\infty}\|f-H_{2n-1}(f)\|_1=0$;

(b) 若 $f\in C^k_{[-1,1]}(k\geqslant3)$, 则 $\|f-H_{2n-1}(f)\|_1=O\left(\frac{1}{n^{k-2}}\right)_{(n\to\infty)}$;

(c) 若 $f\in C^k_{[-1,1]}(k\geqslant2)$, 且 $f^{(k)}\in Lip\alpha$ $(0<\alpha\leqslant1)$.

则

$$\|f-H_{2n-1}(f)\|_1=O\left(\frac{1}{n^{k+\alpha-2}}\right)_{(n\to\infty)}.$$

人们自然会问：是否存在这样的节点组X_n，对于它来说，使（1）成立的函数类比$C^2_{[-1,1]}$更大? 对此，本文将作出肯定的回答．事实上，我们将证明取$w(x)=(1-x^2)U_n(x)$的零点为节点时，有 $\|H_{2n+3}\|_1=O(\ln n)$，其中 $U_n(x)=\frac{\sin(n+1)\theta}{\sin\theta}(x=\cos\theta)$ 是第二类 Чебышев 多项式，算子 H_{2n+3} 的范数 $\|H_{2n+3}\|_1$ 为：

$$\|H_{2n+3}\|_1=\sup_{\substack{f\in C^1_{[-1,1]}\\ \|f\|_1\leqslant1}}\{\|H_{2n+3}(f;x)\|_1\}.$$

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因此，对导数满足 Dini-Lipschitz 条件的函数，(1) 式都成立。

下文，节点组 X_{n+2} 均取 $w(x)=(1-x^2)U_n(x)$ 的零点

$$x_{k;n+2}=\cos\theta_{k,n+2}=\cos\frac{k\pi}{n+1},\ k=0,1,2,\cdots,n,\ n+1,\ n=1,2,\cdots. \tag{2}$$

为简便计，记 $x_k\equiv x_{k,n+2}$，$k=0,1,\cdots,n+1$.

我们知道，对任一 $f\in C^1_{[-1,1]}$，满足

$$\begin{cases} H_{2n+3}(f;x_k)=f(x_k), \\ H'_{2n+3}(f;x_k)=f'(x_k). \end{cases} \qquad k=0,1,\cdots,n+1$$

的 $2n+3$ 阶 Hermite 插值多项式 $H_{2n+3}(f;x)$ 可写成：$H_{2n+3}(f;x)=\sum\limits_{k=0}^{n+1}f(x_k)v_k(x)l_k^2(x)$ $+\sum\limits_{k=0}^{n+1}f'(x_k)(x-x_k)l_k^2(x)$,

其中

$$l_k(x)=\frac{w(x)}{w'(x_k)(x-x_k)},\qquad k=0,1,\cdots,n+1;$$

$$v_k(x)=1-\frac{w''(x_k)}{w'(x_k)}(x-x_k),\qquad k=0,1,\cdots,n+1.$$

由于 $w(x)=(1-x^2)U_n(x)$，经简单计算可得

$$l_k(x)=\begin{cases} (-1)^{k+1}\dfrac{(1-x^2)U_n(x)}{(n+1)(x-x_k)} & (k=1,2,\cdots,n); \\ \dfrac{(1+x)U_n(x)}{2(n+1)} & (k=0); \\ (-1)^n\dfrac{(1-x)\,U_n(x)}{2(n+1)} & (k=n+1) \end{cases} \tag{3}$$

以及

$$v_k(x)=\begin{cases} 1+\dfrac{x_k}{1-x_k^2}\quad (x-x_k)(k=1,2,\cdots,n); \\ 1\mp\dfrac{1}{3}(2n^2+4n+3)(x\mp1)\left(k=\begin{matrix}0\\n+1\end{matrix}\right). \end{cases} \tag{4}$$

显然 $x=x_k$ 是不必讨论的。所以下文都认定 $x\neq x_k\quad(k=0,1,\cdots,n+1)$.

引理1　1) $(1-x^2)|U_n(x)|\sum\limits_{k=1}^{n}\dfrac{1}{|x-x_k|}=O(n\ln n)$;

2) $(1-x^2)U_n^2(x)\sum\limits_{k=1}^{n}\dfrac{1}{|x-x_k|}=O(n^2)$.

证明　不妨设 $x>0$.

1) $$(1-x^2)|U_n(x)|\sum_{k=1}^{n}\frac{1}{|x-x_k|}=\sum_{k=1}^{n}\frac{\sin\theta\,|\sin(n+1)(\theta-\theta_k)|}{2\sin\frac{\theta+\theta_k}{2}\,\sin\frac{\theta-\theta_k}{2}}$$

$$= \sum_{|\theta-\theta_k|\le\frac{\pi}{2(n+1)}} \cdot + \sum_{|\theta-\theta_k|>\frac{\pi}{2(n+1)}} \cdot$$

$$\le 2(n+1) + \sum_{|\theta-\theta_k|>\frac{\pi}{2(n+1)}} \frac{\pi}{|\theta-\theta_k|}$$

$$\le 2(n+1) + 2\sum_{k=1}^{n} \frac{n}{k} = O(n\ \ln n).$$

这里我们用到两个熟知的不等式：

$$\sin\theta \le 2\sin\frac{\theta+\theta_k}{2} \qquad (k=1,2,\cdots,n;\ \ 0\le\theta\le\pi)$$

与

$$|\sin nt| \le n|\sin t|.$$

2) 由 $|x-x_k| = 2\sin\frac{\theta+\theta_k}{2}\sin\left|\frac{\theta-\theta_k}{2}\right| \sim |\theta^2-\theta_k^2|$ 即得

$$(1-x^2)U_n^2(x)\sum_{k=1}^{n}\frac{1}{|x-x_k|} = \sum_{k=1}^{n}\frac{\sin(n+1)(\theta+\theta_k)\sin(n+1)(\theta-\theta_k)}{2\sin\frac{\theta+\theta_k}{2}\sin\left|\frac{\theta-\theta_k}{2}\right|}$$

$$= \sum_{|\theta\pm\theta_k|\le\frac{\pi}{2(n+1)}} \cdot + \sum_{|\theta\pm\theta_k|>\frac{\pi}{2(n+1)}} \cdot$$

$$\le O(n^2) + O(1)\sum_{k=1}^{n}\frac{n^2}{k^2} = O(n^2).$$

引理2 $\sum_{k=0}^{n+1}|v_k(x)|\,l_k^2(x) = O(1) \qquad (n\to\infty).$

证明 由(3)、(4)得

$$\sum_{k=0}^{n+1}|v_k(x)|\,l_k^2(x) \le \sum_{k=0}^{n+1} l_k^2(x) + \frac{2n^2+4n+3}{6(n+1)^2}(1-x^2)U_n^2(x)$$

$$+\frac{1}{(n+1)^2}\sum_{k=1}^{n}\left|\frac{x_k(1-x^2)^2U_n^2(x)}{(1-x_k^2)(x-x_k)}\right| \qquad (5)$$

$$= I_1 + I_2 + I_3.$$

由恒等式 $\quad \sum_{k=1}^{n}\frac{1}{x-x_k} = \frac{U_n'(x)}{U(x)} \quad$ 即得

$$\sum_{k=1}^{n}\frac{1}{(x-x_k)^2} = \frac{(U_n'(x))^2 - U_n''(x)\,U_n(x)}{U_n^2(x)}$$

$$= \frac{(n+1)^2 + (n+1)xU_n(x)T_{n+1}(x) - (1+x^2)\,U_n^2(x)}{(1-x^2)^2U_n^2(x)}. \qquad (6)$$

这里，我们用到

$$U_n'(x)=\frac{1}{1-x^2}[xU_n(x)-(n+1)T_{n+1}(x)] \tag{7}$$

以及 $U_n(x)$ 满足微分方程(参见[4,(4,2,1)])

$$(1-x^2)U_n''(x)-3xU_n'(x)+n(n+2)U_n(x)=0.$$

于是，由简单计算而得

$$I_1=\sum_{k=0}^{n+1} l_k^2(x)=\frac{(1+x^2)U_n^2(x)}{2(n+1)^2}+\frac{(1-x^2)^2U_n^2(x)}{(n+1)^2}\sum_{k=1}^{n}\frac{1}{(x-x_k)^2}=O(1). \tag{8}$$

显然，

$$I_3=\frac{1}{(n+1)^2}\sum_{k=1}^{n}\left|\frac{x_k(1-x^2)^2U_n^2(x)}{(1-x_k^2)(x-x_k)}\right|=\frac{(1-x^2)U_n^2(x)}{(n+1)^2}\sum_{k=1}^{n}\left|\frac{x_k}{x-x_k}-\frac{(x+x_k)x_k}{1-x_k^2}\right|$$

$$\leqslant\frac{(1-x^2)U_n^2(x)}{(n+1)^2}\sum_{k=1}^{n}\left(\frac{1}{|x-x_k|}+\frac{2}{1-x_k^2}\right).$$

但是

$$\sum_{k=1}^{n}\frac{1}{1+x_k}=\sum_{k=1}^{n}\frac{1}{1-x_k}=\frac{U_n'(1)}{U_n(1)}=\frac{n(n+2)}{3},$$

从而

$$\sum_{k=1}^{n}\frac{1}{1-x_k^2}=\frac{n(n+2)}{3}. \tag{9}$$

于是由此与引理1得

$$I_3=O(1).$$

将此式与(5)、(8)以及显然的估计

$$I_2=\frac{2n^2+4n+3}{6(n+1)^2}(1-x^2)U_n^2(x)=O(1)$$

结合，就完成了引理2的证明.

定理1 $\|H_{2n+3}\|_1=O(\ln n)\qquad(n\to\infty)$.

证明 今考察任一满足 $\|f\|_1=1$ 的函数 $f\in C^1_{[-1,1]}$.由前述引理可见

$$\|H_{2n+3}\|_C=O(1)\qquad(n\to\infty), \tag{10}$$

其中 $\|\cdot\|_C$ 是 $C_{[-1,1]}$，空间里的范数.

对于 $f\in C^1_{[-1;1]}$，显然有

$$H_{2n+3}(f;\ x)-f(x)=\sum_{k=0}^{n+1}\nu_k(x)l_k^2(x)(f(x_k)-f(x))+\sum_{k=0}^{n+1}f'(x_k)(x-x_k)l_k^2(x)$$

$$=\sum_{k=0}^{n+1}\nu_k(x)l_k^2(x)\int_x^{x_k}f'(t)dt+\sum_{k=0}^{n+1}f'(x_k)(x-x_k)l_k^2(x).$$

微分上式得

$$H'_{2n+3}(f;\ x)=\sum_{k=0}^{n+1}\nu_k(x)l_k^2(x)\int_x^{x_k}f'(t)dt+2\sum_{k=0}^{n+1}\nu_k(x)l_k(x)l'_k(x)\int_x^{x_k}f'(t)dt$$

$$+2\sum_{k=0}^{n+1}f'(x_k)(x-x_k)l_k(x)l'_k(x)+\sum_{k=0}^{n+1}f'(x_k)l_k^2(x). \tag{11}$$

依次记右边四项为 I_4，$2I_5$，$2I_6$ 和 I_7，则由(3)、(4)、(9)与引理 1 得出

$$|I_4|\leqslant\frac{2n^2+4n+3}{6(n+1)^2}(1-x^2)U_n^2(x)+\frac{(1-x^2)U_n^2(x)}{(n+1)^2}\sum_{k=1}^{n}\left(\frac{1}{|x-x_k|}+\frac{2}{1-x_k^2}\right)$$

$$=O(1). \tag{12}$$

由(8)，

$$|I_7|\leqslant\sum_{k=0}^{n+1}l_k^2(x)=O(1). \tag{13}$$

由于

$$\begin{cases} l'_k(x)=\dfrac{(-1)^k[xU_n(x)+(n+1)T_{n+1}(x)]}{(n+1)(x-x_k)}-\dfrac{l_k(x)}{x-x_k},\quad k=1,2,\cdots,n; \\ l'_0(x)=\dfrac{(1+x)U_n(x)+U_n(x)}{2(n+1)}; \\ l'_{n+1}(x)=\dfrac{(-1)^n}{2(n+1)}[-U_n(x)+(1-x)U'_n(x)], \end{cases} \tag{14}$$

故由 (3)，(7)，(8) 与 $\|f\|_1=1$ 有

$$I_6=\frac{-(1-x^2)U_n(x)}{4(n+1)^2}f'(1)[(1+x)U'_n(x)+U_n(x)]$$

$$+\frac{(1-x^2)U_n(x)}{4(n+1)^2}f'(-1)[(1-x)U'_n(x)-U_n(x)]$$

$$-\frac{(1-x^2)U_n(x)}{(n+1)^2}[xU_n(x)+(n+1)T_{n+1}(x)]\sum_{k=1}^{n}\frac{f'(x_k)}{x-x_k}$$

$$-\sum_{k=1}^{n}f'(x_k)l_k^2(x)$$

$$=O(1)-\frac{(1-x^2)U_n(x)}{(n+1)^2}[xU_n(x)+(n+1)T_{n+1}(x)]\sum_{k=1}^{n}\frac{f'(x_k)}{x-x_k}.$$

利用引理 1 就得到

$$I_6=O(\ln n). \tag{15}$$

由(3),(4),(14)整理可得

$$I_5=\sum_{k=0}^{n+1}v_k(x)l_k(x)l_k'(x)\int_x^{x_k}f'(t)dt$$

$$=\frac{U_n(x)}{4(n+1)^2}\Big\{[(1+x)U_n(x)+(1+x)^2U_n'(x)]\int_x^1 f'(t)dt$$

$$+[(1-x)^2U_n'(x)-(1-x)U_n(x)]\int_x^{-1}f'(t)\,dt\Big\}$$

$$+\frac{(2n^2+4n+3)}{12(n+1)^2}\Big\{(1-x^2)[\int_x^1 f'(t)dt-\int_x^{-1}f'(t)dt]$$

$$+x[(1+x)\int_x^1 f'(t)dt+(1-x)\int_x^{-1}f'(t)dt]\Big\}$$

$$-\Big\{\frac{2n^2+4n+3}{12(n+1)}U_n(x)T_{n+1}(x)[(1+x)\int_x^1 f'(t)dt+(1-x)\int_x^{-1}f'(t)dt]$$

$$+\frac{(1-x^2)U_n(x)T_{n+1}(x)}{n+1}\sum_{k=1}^{n}\frac{x_k}{(1-x_k^2)(x-x_k)}\int_x^{x_k}f'(t)dt\Big\}$$

$$-\frac{x(1-x^2)U_n^2(x)}{(n+1)^2}\sum_{k=1}^{n}\frac{1}{(x-x_k)^2}\int_x^{x_k}f'(t)dt$$

$$-\frac{(1-x^2)U_n(x)T_{n+1}(x)}{n+1}\sum_{k=1}^{n}\frac{1}{(x-x_k)^2}\int_x^{x_k}f'(t)dt$$

$$-\frac{x(1-x^2)U_n^2(x)}{(n+1)^2}\sum_{k=1}^{n}\frac{x_k}{(1-x_k^2)(x-x_k)}\int_x^{x_k}f'(t)dt$$

$$-\frac{(1-x^2)^2U_n^2(x)}{(n+1)^2}\sum_{k=1}^{n}\frac{1}{(x-x_k)^3}\int_x^{x_k}f'(t)dt$$

$$-\frac{(1-x^2)^2U_n^2(x)}{(n+1)^2}\sum_{k=1}^{n}\frac{x_k}{(1-x_k^2)(x-x_k)^2}\int_x^{x_k}f'(t)dt,$$

我们把等号右边的项依次记为 I_{51}, I_{52}, $\cdots$, I_{58}. 显然有

$$I_{51}=\frac{U_n(x)}{4(n+1)^2}\Big\{\Big[(1+x)U_n(x)+(1+x)^2U_n'(x)\Big]\int_x^1 f'(t)dt$$

$$+\Big[(1-x)^2U_n'(x)-(1-x)U_n(x)\Big]\int_x^{-1}f'(t)dt\Big\}=O(1).$$

同样有 $I_{52}=O(1)$ 与 $I_{56}=O(1)$。由引理 1 易见

$$|I_{55}| = \left|\frac{(1-x^2)U_n(x)T_{n+1}(x)}{n+1}\sum_{k=1}^{n}\frac{1}{(x-x_k)^2}\int_x^{x_k}f'(t)dt\right|$$

$$\leqslant \frac{(1-x^2)|U_n(x)|}{n+1}\sum_{k=1}^{n}\frac{1}{|x-x_k|}=O(\ln n).$$

类似地可得

$$|I_{54}|=O(1).$$

由(6)式

$$|I_{57}|\leqslant\frac{(1-x^2)^2U_n^2(x)}{(n+1)^2}\sum_{k=1}^{n}\frac{1}{(x-x_k)^2}$$

$$=\frac{1}{(n+1)^2}\{(n+1)^2+(n+1)xU_n(x)T_{n+1}(x)-(1+x^2)U_n^2(x)\}$$

$$=O(1).$$

至于 I_{58}，由于

$$|I_{58}|=\frac{(1-x^2)U_n^2(x)}{(n+1)^2}\left|\sum_{k=1}^{n}\left(\frac{1}{x-x_k}-\frac{x+x_k}{1-x_k^2}\right)\frac{x_k}{x-x_k}\int_x^{x_k}f'(t)dt\right|$$

$$\leqslant\frac{(1-x^2)U_n^2(x)}{(n+1)^2}\sum_{k=1}^{n}\left(\frac{1}{|x-x_k|}+\frac{2}{1-x_k^2}\right),$$

再次利用引理 1 及(9)即得 $I_{58}=O(1)$。

最后，

$$-I_{53}=\frac{U_n(x)T_{n+1}(x)}{n+1}\left\{\frac{2n^2+4n+3}{12}\left[(1+x)\int_x^1 f'(t)dt+(1-x)\int_x^{-1}f'(t)dt\right]\right.$$

$$\left.+(1-x^2)\sum_{k=1}^{n}\frac{x_k}{(1-x_k^2)(x-x_k)}\left[f(x_k)-f(x)\right]\right\}$$

$$=\frac{U_n(x)T_{n+1}(x)}{n+1}\left\{\frac{2n^2+4n+3}{12}\left[(1+x)\int_x^1 f'(t)dt+(1-x)\int_x^{-1}f'(t)dt\right]\right.$$

$$\left.+x\sum_{k=1}^{n}\frac{f(x_k)-f(x)}{x-x_k}-\sum_{k=1}^{n}\frac{f(x_k)}{1-x_k^2}+f(x)\sum_{k=1}^{n}\frac{1}{1-x_k^2}-x\sum_{k=1}^{n}\frac{x_kf(x_k)}{1-x_k^2}\right\}$$

$$=\frac{U_n(x)T_{n+1}(x)}{n+1}\left\{\frac{1}{4}\left[(1+x)(f(1)-f(x))+(1-x)(f(-1)-f(x)\right]\right.$$

$$+\frac{1}{2}\left[(1+x)\sum_{k=1}^{n}\frac{f(1)-f(x_k)}{1-x_k}+(1-x)\sum_{k=1}^{n}\frac{f(-1)-f(x_k)}{1+x_k}\right]$$

$$-x\sum_{k=1}^{n}\frac{f(x)-f(x_k)}{x-x_k}\Bigg\}$$

$$=\frac{U_n(x)T_{n+1}(x)}{2(n+1)}\left\{(1+x)\sum_{k=1}^{n}\left[\frac{f(1)-f(x_k)}{1-x_k}-\frac{f(x)-f(x_k)}{x-x_k}\right]\right.$$

$$\left.+(1-x)\sum_{k=1}^{n}\left[\frac{f(-1)-f(x_k)}{1+x_k}+\frac{f(x)-f(x_k)}{x-x_k}\right]\right\}+O(1).$$

由于

$$\frac{f(1)-f(x_k)}{1-x_k}-\frac{f(x)-f(x_k)}{x-x_k}=\frac{(1-x_k)(f(1)-f(x))-(1-x)(f(1)-f(x_k))}{(1-x_k)(x-x_k)}$$

$$=O\left(\frac{1-x}{|x-x_k|}\right),$$

且

$$\frac{f(-1)-f(x_k)}{1+x_k}+\frac{f(x)-f(x_k)}{x-x_k}=O\left(\frac{1+x}{|x-x_k|}\right),$$

所以

$$|I_{53}|=O(1)\frac{(1-x^2)\,|U_n(x)-T_{n+1}(x)|}{n+1}\sum_{k=1}^{n}\frac{1}{|x-x_k|}=O(\ln n).$$

综上所述，就得 $I_5=O(\ln n)$. 再结合(11)，(12)，(13)与(15)得

$$\|H'_{2n+3}\|_C=O(\ln n).$$

由此与(10)结合，就证明了定理.

注 在定理的证明过程中，我们并未用到 $f'(x)$的连续性，而只用到 $f'(x)$ 的处处存在与有界性.

对于任一 $f\in C^1_{[-1,1]}$，定义

$$E_n(f)=\inf_{p_n\in\mathscr{P}_n}\|f-p_n\|_C,$$

$$E^1_n(f)=\inf_{p_n\in\mathscr{P}_n}\|f-p_n\|_1,$$

其中 $\mathscr{P}_n$ 是阶$\leqslant n$ 的代数多项式全体. 利用显然的不等式

$$E^1_n(f)\leqslant E_{n-1}(f')$$

与

$$\|f-H_{2n+3}(f)\|_1\leqslant(\|H_{2n+3}\|_1+1)E^1_{2n+3}(f),$$

立得

定理2 对于节点组(2)，

(a) 若 $f(x)\in C^{1}_{[-1,1]}$，$f'(x)$满足 Dini-Lipschitz条件：$\lim\limits_{n\to\infty}\omega\left(f',\frac{1}{n}\right)\ln n=0$，则有

$$\lim_{n\to\infty}\|f(x)-H_{2n+3}(f;x)\|_1=0;$$

(b) 若 $f(x)\in C^{k}_{[-1,1]}(k\geqslant 1)$，则有

$$\|f(x)-H_{2n+3}(f;x)\|_1=o\left(\frac{\ln n}{n^{k-1}}\right)\qquad(n\to\infty);$$

(c) 若$f(x)\in C^{k}_{[-1,1]}(k\geqslant 1)$，且$f^{(k)}(x)\in \mathrm{Lip}\,\alpha\ (0<\alpha\leqslant 1)$，则有

$$\|f(x)-H_{2n+3}(f;x)\|_1=o\left(\frac{\ln n}{n^{k+\alpha-1}}\right)\qquad(n\to\infty).$$

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On the Approximations of Functions and Their Derivatives by Hermite Interpolation

By Yu Dingguo (郁定国)

Abstract

Given the space $C^{k}_{[-1,1]}$ consisting of k-times continuonsly differentiable real-valued function. Further, we provide $C^{k}_{[-1,1]}$ with the norm $\|f\|_k$, which for a given $f\in C^{k}_{[-1,1]}$ is defined by

$$\|f\|_k=\max_{0\leqslant\mu\leqslant k}\left(\sup_{x\in[-1,1]}|f^{(\mu)}(x)|\right)$$

Let(with $x_{kn}\equiv x_k$)X_n: $-1\leqslant x_n<x_{n-1}<\cdots<x_1\leqslant 1$, ($n=1, 2,\cdots$) denote an arbitrary nodal matrix. we consider the Hermite interpolation operator

$$H_{2n-1}:\ C^1_{[-1\ 1]}\to C^1_{[-1\ 1]}$$

It is known that the convergence

$$\lim_{n\to\infty}\|f-H_{2n-1}f\|_1=0$$

does not hold for each $f\in C^1_{[1-,1]}$ (cf. [1], [2]).

If X_n is sequence of the Tchbycheff nodes of the first kind, P. Pottinger[3]. has proved:

Theorem P. (a) For a given $f\in C^2_{[-1,1]}$ we have $\lim\limits_{n\to\infty}\|f-H_{2n-1}f\|_1=0$;

(b) If $f\in C^k_{[-1,1]}(k\geqslant 3)$, we get $\|f-H_{2n-1}f\|_1=O\left(\frac{1}{n^{k-2}}\right)$ $(n\to\infty)$;

(c) For $f\in C^k_{[-1,1]}(k\geqslant 2)$ with $f^{(k)}\in$ Lip α $(o<\alpha\leqslant 1)$, we obtain

$$\|f-H_{2n-1}f\|_1=O\left(\frac{1}{n^{k+\alpha-2}}\right)\qquad(n\to\infty).$$

For X_{n+2} is sequence of the roots of $w(x)=(1-x^2)U_n(x)$, where $U_n(x)=\frac{\sin(n+1)\theta}{\sin\theta}$ $(x=\cos\theta)$ is the Tchebycheff polynomial of the second kind, the main results of this note are the following two theorems;

Theorem 1. $\|H_{2n+3}\|_1=\sup\limits_{\substack{f\in C^1_{[k-1,1]}\\ \|f\|_1\leqslant 1}}\{\|H_{2n+3}(f;\ x)\|_1\}=O(\ln n)\qquad(n\to\infty).$

Theorem 2. (a)For $f\in C^1_{[-1,1]}$ with $f'(x)$ satisfying Dini-Lipschitz's conditon $\lim\limits_{n\to\infty}\omega\left(f';\ \frac{1}{n}\right)\ln n=0$, then we have

$$\lim_{n\to\infty}\|f-H_{2n+3}f\|_1=0;$$

(b) If $f\in C^k_{[-1,1]}$ $k(\geqslant 1)$, then

$$\|f(x)-H_{2n+3}(f;\ x)\|_1=o\left(\frac{\ln n}{n^{k-1}}\right)\qquad(n\to\infty);$$

(c) For $f\in C^k_{[-1,1]}(k\geqslant 1)$with $f^{(k)}\in$ Lip α $(0<\alpha\leqslant 1)$ we obtain

$$\|f(x)-H_{2n+3}(f;\ x)\|_1=O\left(\frac{\ln n}{n^{k+\alpha-1}}\right)\qquad(n\to\infty).$$