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A NOTE ON MINIMAL RINGS AND FIELDS*

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The purpose of this note is to give some simple description of the types and numbers of sets that are contained respectively in the minimal ring, σ-ring, field, and σ-field generated by a given class of sets. A few applications are also given.

In set or measure theory, the following facts are usually introduced at the very beginning: Let $\mathbf{A}$ be a class of sets in a given space S, then there exists a unique ring $\mathbf{R}(\mathbf{A})$ such that $\mathbf{A}\subseteq\mathbf{R}(\mathbf{A})$; and if $\mathbf{R}$ is any ring containing $\mathbf{A}$, then $\mathbf{R}(\mathbf{A})\subseteq\mathbf{R}$ ([2], p. 22, Theorem A). The ring $\mathbf{R}(\mathbf{A})$ is said to be generated by $\mathbf{A}$, and is called the minimal ring containing $\mathbf{A}$. Similarly, the minimal σ-ring $\mathbf{R}_\sigma(\mathbf{A})$, field $\mathbf{F}(\mathbf{A})$, and σ-field $\mathbf{F}_\sigma(\mathbf{A})$ containing $\mathbf{A}$ can also be defined and shown to exist ([2], p. 24). Recall that a ring (field) is a non-empty class of sets closed with respect to the formation of finite unions and differences (complements). If the word "finite" is replaced by the word "countable," then we have the definitions for σ-ring and σ-field respectively.

Students interested in concrete demonstrations often like to ask the following questions:

What types of sets are contained in $\mathbf{R}(\mathbf{A})$, $\mathbf{R}_\sigma(\mathbf{A})$, etc.?

How many sets does each of them contain?

One type of answer to the first question was given by Cramér ([1], p. 14) as follows. The class $\mathbf{R}_\sigma(\mathbf{A})$, for example, is the totality of all sets that can be ob-

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tained by performing on countable numbers of sets in **A**, the operations union, intersection, and subtraction a countable number of times (see also [2], p. 26, (9)). The statement, unfortunately, is somewhat involved and does not provide a clear picture. There are other ways to characterize the said classes. Halmos, for example, showed that every set in **R**(**A**) may be covered by a finite union of sets in **A**([2], p. 22, Theorem B.). As for the second question, Halmos ([2], p. 23, Theorem C) showed that if **A** is countable, then so is **R**(**A**). More generally, he ([2], p. 26, (8)) stated that **A** and **R**(**A**) have the same cardinal number. The purpose of this note is to show that both questions can be answered in a way which is more concrete and easier to understand.

Let **A** be a given class of sets. For any A in **A**, define $A^i = A$ if $i = 0$ and $A^i = A'$ if $i = 1$, where A' is the complement of A. Now construct a class of sets in the following way:

(a) Obtain all sets of the form $\prod_{k=1}^{n} A_{t_k}^{i_k}$, where Π denotes intersection, $A_{t_k} \in \mathbf{A}$, $i_k = 0$ or 1, $k = 1, 2, \cdots, n$, and $n = 1, 2, \cdots$.

(b) For every $n = 1, 2, \cdots$, delete those whose exponents $i_k = 1$, for all $k = 1, 2, \cdots, n$,

(c) Obtain all sets each of which is the union of a finite number of sets in (a) that are not deleted.

It is easy to see that if $\mathbf{C}_1$ is the class of all sets in (c), then $\mathbf{C}_1 = \mathbf{R}(\mathbf{A})$. Furthermore, if step (b) is omitted, i.e., no deletion is made, and the corresponding class of all sets in (c) is denoled by **C**, then $\mathbf{C} = \mathbf{F}(\mathbf{A})$. The classes $\mathbf{R}_\sigma(\mathbf{A})$ and $\mathbf{F}_\sigma(\mathbf{A})$ are more difficult to characterize and we do not have a complete answer. Let $\mathbf{D}_1$ and $\mathbf{E}_1$ be constructed in the same way as $\mathbf{C}_1$ except that countable and non-countable intersections and unions are allowed respectively. Let **D** and **E** be the corresponding classes constructed without deletion (b). Since $\mathbf{R}_\sigma(\mathbf{A})$ is a monotone class ([2], p. 27, Theorem A) and contains **R**(**A**), we see that $\prod_{k=1}^{\infty} A_{t_k}^{i_k} = \operatorname{Lim}_{n\to\infty} \prod_{k=1}^{n} A_{t_k}^{i_k}$ is in $\mathbf{R}_\sigma(\mathbf{A})$. Hence $\mathbf{D}_1 \leqslant \mathbf{R}_\sigma(\mathbf{A})$. In a similar way, we establish

Theorem 1. Let **A** be a given class of sets. Then $\mathbf{R}(\mathbf{A}) = \mathbf{C}_1 \leqslant \mathbf{D}_1 \leqslant \mathbf{R}_\sigma(\mathbf{A}) \leqslant \mathbf{E}_1$, and $\mathbf{F}(\mathbf{A}) = \mathbf{C} \leqslant \mathbf{D} \leqslant \mathbf{F}_\sigma(\mathbf{A}) \leqslant \mathbf{E}$.

Remark 1. We show by examples that the "inequalities" in the above theorem may hold. Let **A** be the class of all finite and infinite intervals whose end points are rational numbers. Then $F(A)$ is the class of all sets which are the unions of finite numbers of sets in A. If X is the set containing the irrational number x only, then $X \notin F(A)$. But $X \in D$, since $X = \prod_n (a_n, b_n)$, where a_n and b_n are rational and both tend to x as $n \to \infty$. Hence $F(A) \subset D$. Furthermore, it is well known that $F_\sigma(A)$ is the Borel field of sets in the space S of real numbers. The class E is clearly the class of all sets in S. Since non-Borel sets exist ([2], p. 67), we see that $F_\sigma(A) \subset E$. It is not clear, however, whether $C \subset F_\sigma(A)$.

Remark 2. Some of the known facts in Set Theory are immediate consequences of Theorem 1. One example is Theorem 13 in([2], p. 22). Others are ([2], p. 26, (10)): If $E \in R(A)$, then there exists a finite subclass B of A such that $E \in R(B)$. If E is any set in the space where $R(A)$ is defined, then $R(A) \cap E = R(A \cap E)$. The corresponding statements also hold if R is replaced by F.

As to the second question, we give the following answer.

Theorem 2. Let A be a given class of sets. If A contains a finite number k of sets, then $R(A) = R_\sigma(A)$ contains at most 2^{2^k-1} sets, and $F(A) = F_\sigma(A)$ contains at most 2^{2^k} sets. If A contains an infinite number of sets, then A, $R(A)$, and $F(A)$ have the same cardinal number, while $R_\sigma(A)$ and $F_\sigma(A)$ may have larger cardinal numbers.

Proof. Suppose that A is a finite class and contains $A_1, \cdots, A_k$. Then, by Theorem 1, $R(A)$ is the class of all sets which are either empty or finite unions of sets of the form $\prod_{t=1}^{k} A_t^{i_t}$, where $i_t \neq 1$. There are $2^k - 1$ sets of such form, and one or more of them may be included in a particular union of those sets. Hence $R(A)$ contains at most 2^{2^k-1} sets. Now $E_1 = C_1$, hence $R(A) = R_\sigma(A)$. Similarly, $F(A) = F_\sigma(A)$ contains at most 2^{2^k} sets. In general, let A have infinite cardinal number α. For a given n and $i_1, \cdots, i_n$, there are $\alpha^n = \alpha$ sets of the form $\prod_{k=1}^{n} A_{t_k}^{i_k}$, where $A_{t_k} \in A$, $k = 1, \cdots, n$. Hence, the class of all sets defined in (a) has cardinal number $\sum_{n=1}^{\infty} 2^n \alpha = \alpha$; and the class C of all sets in (c) without deletion has cardinal number $\sum_{n=1}^{\infty} \alpha^n = \alpha$ ([3], p. 419). Since $A \subseteq R(A) \subseteq F(A) = C$, they all should have the same cardinad number.

Remark 3. The above proof seems to be somewhat more straightforward than the one given by Halmos ([2], p. 23, Theorem C). Also, as an sxample, let $A =$

$\{A, B\}$. Then $\mathbf{R}(A)=\{O, AB, AB', A'B, A, B, AB'\cup A'B, A\cup B\}$, a total of $2^{2^2-1}=8$ sets. Finally, the example in Remark 1 shows that $\mathbf{R}_\sigma(\mathbf{A})$ and $\mathbf{F}_\sigma(\mathbf{A})$ may have larger cardinal numbers than A.

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