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The Integral of Function in a Class of Schlicht Functions*

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Suppose $f(z)$ is analytic in the unit disc $|z|<1$ with normalization $f(0)=1-f'(0)=0$ and may be written as $f(z)\in A$. Denote a schlicht subclass of A by S. For $\sigma<1$, let

$$S(\sigma)=\{f\,|\,f\in A \text{ and } \mathrm{Re}\frac{zf'}{f}>\sigma;\ |z|<1\},$$

$$K(\sigma)=\{f\,|\,f\in A \text{ and } \mathrm{Re}\left(1+\frac{zf''}{f'}\right)>\sigma;\ |z|<1\},$$

$$C(\sigma)=\{f\,|\,f\in A \text{ and there exists } g\in K(\sigma) \text{ such that } \mathrm{Re}\left\{\frac{f'(z)}{g'(z)}\right\}>0;\ |z|<1\}$$

be the families of functions with normalization σ-order starlike, σ-order convex and σ-order close-to-convex, respectively.

In the case of $\alpha>-1$, let

$$D^\alpha f(z)=\frac{z}{(1-z)^{1+\alpha}}*f(z), \qquad (|z|<1)$$

where "*" is the symbol of Hadamard product of two analytic functions. If $\alpha=n$ is a non-negative integer, we have

$$D^n f(z)=\frac{z(z^{n-1}f(z))^{(n)}}{n!}. \qquad (|z|<1)$$

It is found by computation that

$$z(D^\alpha f)'=(\alpha+1)D^{\alpha+1}f-\alpha D^\alpha f.$$

St. Ruscheweyh has considered the function class K_α:

$$K_\alpha=\left\{f\,|\,f\in A,\ \mathrm{Re}\left(\frac{D^{\alpha+1}f}{D^\alpha f}\right)>\frac{1}{2};\ |z|<1\right\},$$

where $\alpha>-1$. Evidently, $f(z)\in K_\alpha$ $(\alpha>-1)$ if and only if $D^\alpha f\in S\left(\frac{1-\alpha}{2}\right)$.

We now introduce a function class R_α:

$$R_\alpha=\left\{f\,|\,f\in A, \mathrm{Re}\left(\frac{D^{\alpha+1}f}{D^\alpha f}\right)>\frac{\alpha}{1+\alpha};\ |z|<1\right\},$$

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where $\alpha>-1$. It is evident that $f\in R_\alpha(\alpha>-1)$ if and only if $D^\alpha f\in S(0)$, and that $R_\alpha\subseteq K_\alpha$ with $\alpha\geqslant 1$ and $K_\alpha\subseteq R_\alpha$ with $-1<\alpha\leqslant 1$.

In this paper we consider the integral

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt \qquad (|z|<1)$$

where $f(z)\in A$ and $c\neq -1$ is a complex number. S. D. Bernardi[3], St. Ruscheweyh[4], W. Barnard and C. Kellogg[5] have proved under different conditions that if Re $c\geqslant 0$ and $f\in S(0)$, $K(0)$ and $C(0)$, then $F(z)\in S(0)$, $K(0)$ and $C(0)$ respectively. S. D. Bernardi and the authors of [4] and [5] have also considered the inverse problems.

We shall be concerned mainly with generalizing and improving the just-mentioned results. We obtain some theorems as follows:

Theorem 1 Let

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt. \qquad (|z|<1)$$

Then we have

i) If Re $c\geqslant 0$ and $f\in R_\alpha$, then $F(z)\in R_\alpha$.

ii) If $c\geqslant 1$ and $D^\alpha f\in S\left(-\frac{1}{2c}\right)$, or if $0\leqslant c\leqslant 1$ and $D^\alpha f\in S\left(-\frac{c}{2}\right)$, then $F(z)\in R_\alpha$.

iii) If $c=\alpha>-1$ and $f\in R_\alpha$, then $F(z)\in R_{\alpha+1}$.

The proof may be supplied by means of Jack's lemma [7].

In the special cases $\alpha=0$ and $\alpha=1$ respectively, we have

Corollary 1.1 If $c\geqslant 1$ and $f\in S\left(-\frac{1}{2c}\right)$, or if $0\leqslant c\leqslant 1$ aod $f\in S\left(-\frac{c}{2}\right)$, then $F(z)\in S(0)$.

Corollary 1.2 If $c\geqslant 1$ and $f\in K\left(-\frac{1}{2c}\right)$, or if $0\leqslant c\leqslant 1$ and $f\in K\left(-\frac{c}{2}\right)$, then $F(z)\in K(0)$.

It should be noted that if $\sigma<0$, we have $S(\sigma)\supset S(0)$, $K(\sigma)\supset K(0)$. Hence our results have generalized and improved the relevant results in [3], [4] and [5]. For the specific case $c=1$, the result has been obtained in [2].

Since $f\in S$, we have

$$\left|z\frac{f''(z)}{f'(z)}-\frac{2r^2}{1-r^2}\right|\leqslant\frac{4r}{1-r^2} \quad (|z|=r<1).$$

Using corollary 1.2, we get

Corollary 1.3 If $f(z)\in S$, then there is a positive number ρ_c such that $\frac{1}{\rho_c}F(\rho_c z)\in K(0)$ and here

$$\rho_c=\begin{cases}\dfrac{4c-\sqrt{12c^2+1}}{2c-1} & (c\geqslant 1),\\[2mm] \dfrac{4-\sqrt{12+c^2}}{2-c} & (0\leqslant c<1).\end{cases}$$

If $c=1$, the result can be found in [2].

We can also prove the following

Theorem 2 Letting

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \qquad (|z|<1),$$

we have:

i) If $c\geqslant 1$ and $f\in C\left(-\frac{1}{2c}\right)$, then $F(z)\in C(0)$.

ii) If $0<c\leqslant 1$ and $f\in C\left(-\frac{c}{2}\right)$, then $F(z)\in C(0)$.

Using the Yoshikawa-Yoshikai lemma (cf. [8] or [9]), we establish two theorems as follows:

Theorem 3 Suppose $c\neq -1$ is a complex number and

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt$$

or

$$f(z)=\frac{1}{c+1}z^{1-c}(z^cF(z))' \qquad (|z|<1).$$

If $F(z)\in R_\alpha$ $(\alpha>-1)$, then there is a positive constant r_c such that $\frac{1}{r_c}f(r_cz)\in R_\alpha$. Here

$$r_c=\frac{\left(|c|^2+7-\sqrt{14|c|^2+2\mathrm{Re}(c^2)+48}\right)^{\frac{1}{2}}}{|c-1|}.$$

The constant r_c is possibly best and the extremum function $F(z)$ satisfies

$$D^\alpha F(z)=\frac{z}{(1-z)^2} \qquad (|z|<1).$$

In particular, if $c\neq -1$ is a real number, then

$$r_c=\begin{cases}\dfrac{\sqrt{c^2+3}-2}{c-1} & (c>1),\\[2mm] \dfrac{2-\sqrt{c^2+3}}{1-c} & (-1<c\leqslant 1).\end{cases}$$

This result implies the relevant results in [5] and [6]. It is of interest to note that the constant r_c is independent of α.

Theorem 4 Let $c\neq -1$ be a complex number and

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt$$

or

$$f(z)=\frac{1}{c+1}z^{1-c}(z^cF(z))' \qquad (|z|<1).$$

If $F(z)\in K_a$, then there exists a positive constant $r_c(a)$ such that $\frac{1}{r_c(a)}f(r_c(a)z)\in K_a$, where $r_c(a)$ is the smallest positive root of equation

$$|(1+c)-(c-a)r^2|=(a+3)r$$

in (0,1]. The constant $r_c(a)$ is possibly best and the extremum function is $F(z)=\frac{z}{1-z}$.

In particular, if $c\geqslant0$, the last equation is in the form of

$$(c-a)r^2+(a+3)r-(1+c)=0.$$

Hence, it should be noted that theorem 4 is an improvement of the results in [1] and [10], which provides an exact bound $r_c(a)$ for complex number c.

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