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Positive Solutions of Hammerstein Nonlinear Integral Equations*

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In this paper we consider the Hammerstein nonlinear integral equation

$$\varphi(x)=\int_G K(x,y)f(y,\varphi(y))dy=A\varphi(x) \qquad (1)$$

where G is a bounded closed domain in R^n; the function $f(x,u)$ is non-negative, continuous on $G\times[0,+\infty)$ and $f(x,0)\equiv 0$; the kernel $k(x,y)$ is non-negative continuous on $G\times G$. Obviously, A acts in the space $C(G)$ and is completely continuous.

Theorem 1 Suppose that (i) $\lim\limits_{u\to+\infty} u^{-1}f(x,u)=0$ and $\lim\limits_{u\to 0^+} u^{-1}f(x,u)=+\infty$ uniformly for $x\in G$; (ii) there exist $\lambda>0$, $\psi(x)\geqslant 0$, $\psi(x)\not\equiv 0$ such that $\lambda\psi(x)=\int_G K(x,y)\psi(y)dy$. Then (1) has a positive solution.

Proof Let $P=\{\varphi|\varphi\in C(G),\ \varphi(x)\geqslant 0\}$, then P is a cone in $C(G)$ and A acts in the cone P; By (i), there exists a number $r>0$ such that $f(x,u)\geqslant\lambda^{-1}u$ for $0\leqslant u\leqslant r$. Assume that there exist $\varphi_1\in S_r=P\cap\{\varphi|\,\|\varphi\|=r\}$ and $t_1>0$ such that $\varphi_1-A\varphi_1=t_1\psi$, hence $\varphi_1\geqslant t_1\psi$ since $A\varphi_1\in P$; let $t_2=\max\{t|\varphi_1\geqslant t\psi\}$, then $\varphi_1\geqslant t_2\psi$. By (ii), we have

$$\varphi_1=A\varphi_1+t_1\psi=\int_G K(x,y)f(y,\varphi_1(y))dy+t_1\psi(x)$$

$$\geqslant\lambda^{-1}\int_G K(x,y)\varphi_1(y)dy+t_1\psi(x)\geqslant\lambda^{-1}t_2\int_G K(x,y)\psi(y)dy+t_1\psi(x)=(t_1+t_2)\psi(x)$$

i.e., $\varphi_1(x)\geqslant(t_1+t_2)\psi(x)$, in contradiction with $t_2=\max\{t|\varphi_1\geqslant t\psi\}$, thus $\varphi-A\varphi\neq t\psi$ for $\varphi\in S_r$ and $t>0$; without loss of generality we can assume that $\varphi-A\varphi\neq 0$ for $\varphi\in S_r$; By theorem 33.3 in [2], we have $\gamma(I-A,\ V_r(P))=0$, where $V_r(P)=P\cap\{\varphi|\,\|\varphi\|\leqslant r\}$; By (i), it is easy to prove that there exists a number $\mu>0$ such that $0\leqslant f(x,u)\leqslant(2M\text{ mes }G)^{-1}u+\mu$, where $M=\max\limits_{(x,y)\in G\times G}K(x,y)$; let $R=4M\mu\,\text{mes}G$, then for $\varphi\in S_R=P\cap\{\varphi|\,\|\varphi\|=R\}$, we have

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$$\|A\varphi\| = \max_{x\in G}\left|\int_G K(x,y)f(y,\varphi(y))dy\right| \leqslant M\mathrm{mes}G[(2M\mathrm{mes}G)^{-1}R+\mu]$$

$$\leqslant \frac{3}{4}R < R = \|\varphi\|$$

By theorem 33.1 in [2], we have $\gamma(I-A, V_R(P))=1$. It follows that there exists $\varphi^*\in V_R(P)\backslash V_r(P)$ such that $A\varphi^*=\varphi^*$, this completes the proof of the theorem.

Theorem 2 Suppose that (i) $\lim\limits_{u\to+\infty} u^{-1}f(x,u)=0$ and $\lim\limits_{u\to0^+} u^{-1}f(x,u)=+\infty$ uniformly for $x\in G$; (ii) there exists a bounded closed domain $G^*\subset G$, such that $\int_{G^*}K(x,y)dx>0$ $(\forall y\in G^*)$. Then (1) has a positive solution.

The proof of this theorem is similar to that of theorem 1.

Theorem 3 Suppose that (i) $\lim\limits_{u\to+\infty} u^{-1}f(x,u)=+\infty$ and $\lim\limits_{u\to0^+} u^{-1}f(x,u)=0$ uniformly for $x\in G$; (ii) $\int_G K(x,y)dx>0$ $(\forall y\in G)$. Then (1) has a positive solution.

Proof let $P=\{\varphi\,|\,\varphi\in C(G);\varphi(x)\geqslant0,\|\varphi\|_L\geqslant\beta M^{-1}\|\varphi\|_C\}$, where $\beta=\min\limits_{y\in G}\int_G K(x,y)dx>0$, $M=\max\limits_{(x,y)\in G\times G} K(x,y)$. A simple computation shows that P is a cone and A acts in the cone P. Let $N=\beta^{-2}M\mathrm{mes}G$, by (ii), it is easy to see that there exists a number b such that $f(x,u)\geqslant(N+1)u+b$ for $u\geqslant0$. Obviously, there exists a number $u^*>0$ such that $(N+1)u+b\mathrm{mes}G\geqslant Nu$ for $u\geqslant u^*$, let $R=M\beta^{-1}u^*$, then for $\varphi\in S_R=P\cap\{\varphi\,|\,\|\varphi\|=R\}$, we have

$$\mathrm{mes}G\|A\varphi\|_C\geqslant\int_G A\varphi dx\geqslant\beta\int_G f(y,\varphi(y))dy\geqslant\beta[(N+1)\int_G\varphi(y)dy+b\mathrm{mes}G]$$

$$\geqslant\beta[(N+1)\beta M^{-1}R+b\mathrm{mes}G]\geqslant\beta(N\beta M^{-1}R)=R\mathrm{mes}G$$

i.e., $\|A\varphi\|\geqslant\|\varphi\|$ for $\varphi\in S_R$. By (i), there exists a number $r>0$ such that $f(x,\mu)\leqslant(M\mathrm{mes}G)^{-1}u$ for $0\leqslant\mu\leqslant r$; then for $\varphi\in S_r=P\cap\{\varphi\,|\,\|\varphi\|=r\}$, we have

$$\|A\varphi\|\leqslant M\int_G f(y,\varphi(y))dy\leqslant M\mathrm{mes}G(M\mathrm{mes}G)^{-1}r=\|\varphi\|.$$

By theorem 4.7 in [3], it follows that there exists $\varphi^*\in P\cap\{\varphi\,|\,r\leqslant\|\varphi\|\leqslant R\}$ such that $\varphi^*=A\varphi^*$. This completes the proof of the theorem.

Remark The results of this paper have improved the main results in [1];

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