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Generalized Möbius Inversion Theory Associated with Non-Standard Analysis*

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This brief note announces a comprehensive result that includes as particular cases not only the extended Möbius-Rota inversion formulas (cf.[1]) but also some classic inverse relations in analysis (e. g. the fundamental theorem for integral calculus and a kind of Fredholm convolution integral transforms with variable upper limits).

Throughout the note R, $^*R$ and $^*N$ denote respectively the standard, the non-standard real number fields and the set of standard & non-standard natural numbers in $^*R$. Sometimes we have to use several R's or $^*R$'s to denote several different real axes. In particular, frequent use will be made of the most familiar infinite integer ω and the positive infinitesimal $\varepsilon=1/\omega$. We shall need a few definitions as follows.

Definition 1 Let $S\equiv(S;\leqslant)$ be a poset that is not locally finite but consists of a finite number of chains each of which contains all the elements representable by the real points or rational points of an interval or of a union of intervals contained in some R or in some $^*R$. Then S in said to have an ε-partitioned structure $\hat{S}$ whenever all the chains of S have been divided into an infinitude of ε-intervals, so-called the infinitesimal partition with ε-length.

As described above, $\hat{S}$ may be regarded as a poset containing ε-intervals as its elements, if the ordering relation between ε-intervals is assumed to be parallel to that of S. The method of non-standard analysis permits one to define a kind of "incidence algebra" on $\hat{S}\times\hat{S}$, in which the incidence function λ and some related ones (e. g. δ,ζ,μ and others) may be defined similarly as that in the ordinary case. Thus, for instance, if S is totally ordered and if $(x,y)\in\hat{S}\times\hat{S}$ with $x<y$, $y\equiv x$ (mod ε), then we have

$$\lambda^k(x,y)=\binom{(y-x)/\varepsilon-1}{k-1}=\binom{(y-x)\omega-1}{k-1}\quad(\text{in } ^*R)$$

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where the non-standard binomial coefficients can be manipulated in $^*R$ just like that in R. (Note that $x\in\hat{S}$ means $x=\nu\varepsilon$ with $\nu\in{}^*N$.)

Definition 2 Let $\phi(z)=\sum_0^\infty a_k z^k$ $(a_0\neq 0,\ a_k\in{}^*R)$ be a formal power series with its formal inverse $1/\phi(z)=\sum_0^\infty b_k z^k$. Then the pair $\{\mu_1,\mu_2\}$ as defined by

$$\mu_1(x,y)=\sum_{k\geq 0}a_k\lambda^k(x,y),\quad \mu_2(x,y)=\sum_{k\geq 0}b_k\lambda^k(x,y),$$

are called a pair of reciprocal μ-functions with $(x,y)\in\hat{S}\times\hat{S}$. In particular, if S is itself locally finite, one should write $(x,y)\in S\times S$ instead of $(x,y)\in\hat{S}\times\hat{S}$.

Definition 3 Let S be a poset satisfying either of the two conditions: (i) S is locally finite having a minimal element; (ii) S is not locally finite but possesses an ε-partitioned structure $\hat{S}$ in which there is an ε-interval that precedes all other ε-intervals of $\hat{S}$. Then S is said to be a poset having a foremost element.

Sometimes the number sequences $\{a_k\}$ and $\{b_k\}$ occurred in the definition of $\{\mu_1,\mu_2\}$ are called reciprocal sequences in $^*R$.

General Inversion Theorem Let S be a poset having a foremost element. Then for every reciprocal pair of μ-functions $\{\mu_1,\mu_2\}$, we have a pair of inverse relations as follows

$$st(f(y))=st(\sum_{x\leq y}\mu_1(x,y)g(x)),$$
$$st(g(y))=st(\sum_{x\leq y}\mu_2(x,y)f(x)),$$

provided that both the right sides of the formulas have meanings in the non-standard analysis, where f and g are functions of $(S\to{}^*R)$ or of $(\hat{S}\to{}^*R)$, and correspondingly $(x,y)\in S\times S$ or $(x,y)\in\hat{S}\times\hat{S}$, according as S is locally finite or not.

The proof can be accomplished in a similar manner as that for the ordinary case (cf. [1]) and with the aid of non-standard analysis (cf. [2][3]).

Certainly, various choices of $\{\mu_1,\mu_2\}$ may lead to various pairs of inverse relations. In what follows we just mention a few examples in classical analysis.

Example 1 Take $S\equiv[0,\infty)$ with $\hat{S}\equiv\{\varepsilon,2\varepsilon,\cdots,\nu\varepsilon,\cdots\}$ $(\nu\in{}^*N)$. For the ordered set $N\equiv\{1,2,3,\cdots,n\cdots\}$ let us choose a reciprocal pair $\{\mu_1,\mu_2\}$ of the form

$$\mu_1=\varepsilon\zeta=\varepsilon\lambda^0+\varepsilon\lambda,\quad (\lambda^0=\delta)$$
$$\mu_2=\varepsilon^{-1}\zeta^{-1}=\varepsilon^{-1}(\lambda^0-\lambda+\lambda^2-\cdots).$$

Then applying the inversion theorem to $g(x)\in C[0,\infty)$, we have

$$f(n\varepsilon)=\varepsilon\sum_{k=1}^n g(k\varepsilon)$$
$$g(n\varepsilon)=\varepsilon^{-1}\{f(n\varepsilon)-f((n-1)\varepsilon)\},\ f(0)=0.$$

Putting $^*a=n\varepsilon\in{}^*R$ so that $n={}^*a\omega\in{}^*N$ and taking standard parts, we get at once (with $a=st(^*a)$)

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$$f(a)=\int_0^a g(t)dt \Longleftrightarrow g(a)=\frac{d}{da}f(a)=f'(a),\ f(0)=0.$$

This is the fundamental theorem in integral calculus.

Example 2 Consider S and $\hat{S}$ as in Example 1 and take

$$\mu_1=\lambda^0+\varepsilon\lambda,\ \mu_2=\lambda^0-\varepsilon\lambda+\varepsilon^2\lambda^2-\varepsilon^3\lambda^3+\cdots.$$

Then an application of the inversion theorem to $g(x)\in C[0,\infty)$ yields the inverse relations

$$f(a)=g(a)+\int_0^a g(t)dt,$$

$$g(a)=f(a)-\int_0^a e^{-(a-t)}f(t)dt.$$

These are known as Fredholm's type of integral equations with variable upper limits.

Example 3 For the same S and $\hat{S}$ as in the above examples, let $\{\mu_1,\mu_2\}$ be defined by

$$\mu_1=\sum_{k\geq 0}a_k\varepsilon^k\lambda^k,\qquad \mu_2=\sum_{k\geq 0}b_k\varepsilon^k\lambda^k,$$

with $a_0=b_0=1$. Then applying the inversion theorem to continuous functions f and g of $(\hat{S}\to {}^*R)$, we obtain the inverse relations

$$f(a)=g(a)+\sum_{r\geq 1}\frac{a_r}{(r-1)!}\int_0^a(a-t)^{r-1}g(t)dt,$$

$$g(a)=f(a)+\sum_{r\geq 1}\frac{b_r}{(r-1)!}\int_0^a(a-t)^{r-1}f(t)dt,$$

which may be converted into the pair of well-known Fredholm convolution integral transforms

$$f(a)=g(a)+\int_0^a K(a-t)g(t)dt,$$

$$g(a)=f(a)+\int_0^a H(a-t)f(t)dt,$$

where $\{1,K(0),K'(0),K''(0),\cdots\}$ and $\{1,H(0),H'(0),H''(0),\cdots\}$ form a pair of reciprocal sequences, and $K(z)$ and $H(z)$ are assumed to be analytic on the half plane $\mathrm{Re}z\geqslant 0$. (Clearly Ex. 3 implies Ex. 2).

The whole details of this work will appear elsewhere.

References

[1] Hsu, L. C., Jour. of Math. Research & Exposition, (1981), Initial issue, 101-4.

[2] Robinson, A., Non-Standard Analysis, Amsterdam, 1966.

[3] Stroyan, K. D. & Luxemburg, W. A. J., Introduction to the Theory of Infinitesimals, Academic Press, New Yosk, London, 1976.

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