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Convergence Rates of the Distributions of Error Variance Estimates in Linear Models*

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The best possible rate of convergence of the distributions of error variance estimates in linear models, based on the residual sum of squares, is obtained under weakest possible conditions.

1. Introduction

Consider the linear regression model

$$y_j = x_j'\beta + e_j\,,\quad j=1,\cdots,n,\cdots \tag{1}$$

where $\{x_j\}$ is a sequence of known p-vectors, β is an unknown p-vector, $\{e_j\}$ is a sequence of independent random errors, with

$$E(e_j)=0,\ \ \mathrm{Var}(e_j)=\sigma^2,\ \ 0<\sigma^2<\infty,\ \ j=1,2,\cdots. \tag{2}$$

Denote

$$X_n=(x_1\vdots\cdots\vdots x_n),\qquad r_n=\mathrm{rank}(X_n),$$
$$Y_{(n)}=(y_1,\cdots,y_n),\qquad e_{(n)}=(e_1,\cdots,e_n).$$

Using the first n observations in (1), we obtain an estimate of σ^2, based on the residual sum of squares, as follows:

$$\begin{aligned}\hat\sigma_n^2&=\frac{1}{n-r_n}Y_{(n)}'(I_n-X_n'(X_nX_n')^-X_n)Y_{(n)}\\&=\frac{1}{n-r_n}\left\{\sum_{j=1}^n e_j^2-\sum_{j=1}^{r_n}\Big(\sum_{k=1}^n a_{njk}e_k\Big)^2\right\},\end{aligned} \tag{3}$$

with

$$\sum_{k=1}^n a_{nj_1k}a_{nj_2k}=\begin{cases}1, & \text{when } j_1=j_2\\ 0, & \text{otherwise}\end{cases}.$$

Also, $r_n=r\leqq p$ for n sufficiently large, so we can write r instead of r_n in (3) for such n.

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It is well known that $E(\hat{\sigma}_n^2)=\sigma^2$. Standardize $\hat{\sigma}_n^2$ to $Z_n=(\hat{\sigma}_n^2-\sigma^2)/\sqrt{\mathrm{var}(\hat{\sigma}_n^2)}$. It presents no difficulty in proving the asymptotic normality of Z_n under suitable conditions, i. e.,

$$\lim_{n\to\infty}\|G_n-\Phi\|=\lim_{n\to\infty}\{\sup_x|G_n(x)-\Phi(x)|\}=0,$$

where G_n is the distribution of Z_n, and Φ is the standard $N(0,1)$ distribution. More interesting and difficult is the problem of rate of this convergence. This problem was first considered by P. L. Hsu[1], in the special case where $p=1$, $x_1=x_2=\cdots=1$ and $\{e_i\}$ is a sequence of iid. variables with

$$E(e_1)=0,\ 0<\mathrm{Var}(e_1)=\sigma^2<\infty,\ E(e_1^4)>\sigma^4,\ E(e_1^8)<\infty.$$

Hsu obtained the best possible rate of $O(n^{-1/2})$ in this simplest special case of(1); In 1979, Schmidt[3] studied the general case. He obtained a rate $O\ (n^{-m/(2m+2)})$ under the conditions that $\{e_i\}$ is an iid. sequence and $E(e_1^{2m})<\infty$, with $m\geqq 3$ a positive integer.

The purpose of this article is to study this problem under weakest possible conditions. Our result, summarized in the following theorem, gives a complete solution of the problem:

Theorem Suppose that

1. g is an even nonnegative function defined on R^1, $g(x)$ and $x/g(x)$ are non-decreasing for $x\geqq 0$, and $\lim\limits_{x\to\infty} g(x)=\infty$,
2. $\{e_i\}$ is an independent sequence satisfying (2);
3. There exist constants $D_1<\infty$ and $D_2>0$, such that

$$\frac{1}{n}\sum_{i=1}^{n}E\{e_i^4 g(e_i^2)\}\leqq D_1\ , \tag{4}$$

$$\frac{1}{n}\sum_{i=1}^{n}d_i^2\geqq D_2,\ \text{ with } d_i^2=\mathrm{Var}(e_i^2)=E(e_i^4)-\sigma^4. \tag{5}$$

Then

$$\|G_n-\Phi\|=O(1/g(\sqrt{n})), \tag{6}$$

with G_n and Φ defined earlier.

Note that we obtain the Berry-Esseen bound $O(n^{-1/2})$ in the special case $g(x)=|x|$.

2. Some Lemmas

In the following we shall employ the symbol I_A to denote the indicator of A, and C to denote some constant not depending on n but may assume different values on each appearance.

Lemma 1 Suppose that e_1, e_2, $\cdots$ are independent and satisfy the conditions (2), (4), (5). Define

$$\hat{e}_{nj} = e_j I_{(|e_j| \leq n^{1/4})}, \qquad j = 1, \cdots, n$$

$$\zeta_{nj} = \hat{e}^2_{nj} - E(\hat{e}^2_{nj}), \qquad j = 1, \cdots, n$$

$$nB^2_n = \sum_{j=1}^n d^2_j, \quad \sigma^2_{nj} = \mathrm{Var}(\zeta_{nj}), \quad \alpha_{nj} = E\{\zeta^2_{nj} g(\zeta_{nj})\},$$

$$n\tilde{B}^2_n = \sum_{j=1}^n \sigma^2_{nj}, \qquad n\alpha_n = \sum_{j=1}^n \alpha_{nj},$$

Then there exist constants $D_3>0$, $D^0_j>0$, $b_j>0$, $j=1,2,3$, such that

$$B^2_n \leqq D_3, \tag{7}$$

$$|B^2_n - \tilde{B}^2_n| \leqq c/g(\sqrt{n}) \tag{8}$$

$$\tilde{B}^2_n \geqq D^0_2, \qquad \alpha_n \leqq D^0_1 \tag{9}$$

for large n. Further, the set

$$A_n = \{j : \sigma^2_{nj} \geqq b_2, \ \alpha_{nj} \leqq b_3, \ 1 \leqq j \leqq n\} \tag{10}$$

contains at least $b_1 n$ elements.

Proof Choosing suitable $m>0$ such that $g(m)>0$, we have

$$B^2_n \leqq \frac{1}{n}\sum_{j=1}^n E\{e^4_j I_{(e^2_j<m)}\} + \frac{1}{n}\sum_{j=1}^n E\{e^4_j I_{(e^2_j>m)}\}$$

$$\leqq m^2 + \sum_{j=1}^n E\{e^4_j g(e^2_j)\}/ng(m) \leqq m^2 + D_1/g(m).$$

This is (7). Denoting by F_j the distribution of e_j, we have

$$|B^2_n - \tilde{B}^2_n| \leqq \frac{1}{n}\sum_{j=1}^n \{E(e^4_j) - E(\hat{e}^4_{nj})\} + \frac{1}{n}\sum_{j=1}^n \{\sigma^4 - E^2(\hat{e}^2_{nj})\}$$

$$\leqq \frac{1}{n}\sum_{j=1}^n \int_{|x|>n^{1/4}} x^4 dF_j(x) + \frac{2\sigma^2}{n}\sum_{j=1}^n \int_{|x|>n^{1/4}} x^2 dF_j(x)$$

$$\leqq \frac{1}{ng(\sqrt{n})}\sum_{j=1}^n E\{e^4_j g(e^2_j)\} + \frac{2\sigma^2}{n^{3/2} g(\sqrt{n})}\sum_{j=1}^n E\{e^4_j g(e^2_j)\}$$

$$\leqq D_1(1+2\sigma^2)/g(\sqrt{n}),$$

which proves (8). The first assertion of (9) follows from the definition of B^2_n, (5) and (8). For the second, note that

$$|\zeta_{nj}| \leqq \hat{e}^2_{nj} \text{ when } \hat{e}^2_{nj} \geqq E(\hat{e}^2_{nj}), \text{ while } |\zeta_{nj}| \leqq E(\hat{e}^2_{nj}) \leqq \sigma^2$$

when $\hat{e}^2_{nj} < E(\hat{e}^2_{nj})$. Hence

$$\alpha_n = \frac{1}{n}\sum_{j=1}^n E\{\zeta^2_{nj} g(\zeta_{nj}) I_{(\hat{e}^2_{nj} > E\hat{e}^2_{nj})}\} + \frac{1}{n}\sum_{j=1}^n E\{\zeta^2_{nj} g(\zeta_{nj}) I_{(\hat{e}^2_{nj} < E\hat{e}^2_{nj})}\}$$

$$\leqq \frac{1}{n}\sum_{j=1}^n E\{\hat{e}^4_{nj} g(\hat{e}^2_{nj})\} + \frac{1}{n}\sum_{j=1}^n \sigma^4 g(\sigma^2) \leqq D_1 + \sigma^4 g(\sigma^2).$$

Turning to the last assertion of the lemma, we denote by $\#(A)$ the number of elements contained in set A. Putting

$$A_{n1}=\{j:1\leqq j\leqq n,\ \sigma_{nj}^2\geqq D_2^0/2\}$$

then

$$D_2^0\leqq\frac{1}{n}\sum_{j=1}^{n}\sigma_{nj}^2\leqq\frac{1}{2}D_2^0+\frac{1}{n}\sum_{j\in A_{n1}}\sigma_{nj}^2$$

$$\leqq\frac{1}{2}D_2^0+\frac{1}{n}\sum_{j\in A_{n1}}E\{\zeta_{nj}^2 I_{(|\zeta_{nj}|<k)}\}+\frac{1}{ng(k)}\sum_{j\in A_{n1}}\zeta_{nj}^2 g(\zeta_{nj})$$

$$\leqq\frac{1}{2}D_2^0+k^2\#(A_{n1})/n+D_1^0/g(k).$$

Choose $k>0$ such that $g(k)\geqq\frac{3D_1^0}{D_2^0}$. We see that $\#(A_{n1})\geqq\varepsilon_1 n$, with $\varepsilon_1=D_2^0/(6k^2)$. Further, put $\varepsilon_2=\varepsilon_1/2$ and

$$A_{n2}=\{j:1\leqq j\leqq n,\ a_{nj}\leqq D_1^0/\varepsilon_2\}.$$

We see from the second inequality of (9) that $\#(A_{n2})\geqq(1-\varepsilon_2)n$ for sufficiently large integer n; Therefore,

$$\#(A_{n1}\cap A_{n2})\geqq\varepsilon_1 n/2,\ \text{for large } n.$$

So the last assertion follows by taking $b_1=\varepsilon_1/2$, $b_2=D_2^0/2$, and $b_3=2D_1^0/\varepsilon_1$.

Lemma 2 Suppose that the conditions of lemma 1 hold, and $\{a_{nuv}\}$ satisfies

$$\sum_{v=1}^{n}a_{nuv}^2=1,\qquad u=1,\cdots,r;\ n=1,2,\cdots.$$

Then there exists constant L not depending on n or $\{a_{nuv}\}$ such that for n sufficiently large,

$$\#(A_n\cap H_n)\geqq b_1 n/2, \tag{11}$$

where

$$H_n=\{v:1\leqq v\leqq n;\ a_{nuv}^2\leqq L/n\ \text{for}\ u=1,\cdots,r\}.$$

Proof As $\sum_{v=1}^{n}a_{nuv}^2=1$, for any $R>0$ and $u=1,\cdots,r$, we have

$$\#(\{v:\ 1\leqq v\leqq n;\ a_{nuv}^2\geqq R/n\})\leqq n/R.$$

Hence, on putting

$$J_{nR}=\{v:\ 1\leqq v\leqq n;\ a_{nuv}^2\geqq R/n\ \text{for some}\ u\},$$

we have $\#(J_{nR})\leqq rn/R$. Taking $R=2r/b_1=L$ and employing the last conclusion of Lemma 1, one sees that (11) holds for n sufficiently large. Hence the lemma is proved.

In the following i denotes $\sqrt{-1}$.

Lemma 3 Suppose that the conditions of lemma 1 hold; Write $\phi_{nj}(t)$ for the c. f. of ζ_{nj}, There exist $\eta>0$, $\lambda>0$ independent of n such that for each $j\in A_n$ we have

$$\left|\phi_{nj}\left(\frac{t}{\sqrt{n}B_n}\right)\right|\leqq\exp\left(-\lambda t^2/n\right),\ \text{when}\ |t|\leqq g(\sqrt{n})\eta.$$

Proof By definition $|\zeta_{nj}|\leqq\sqrt{n}$, so $|\zeta_{nj}|/g(\zeta_{nj})\leqq\sqrt{n}/g(\sqrt{n})$. Noticing that $B_n\geqq\sqrt{D_2}$, and $\sigma^2_{nj}\geqq b_2$, $a_{nj}\leqq b_3$ for $j\in A_n$, we have

$$\left|\phi_{nj}\left(\frac{t}{\sqrt{n}B_n}\right)-1+\frac{\sigma^2_{nj}}{2nB_n^2}t^2\right|\leqq\frac{|t|^3E|\zeta_{nj}|^3}{n^{3/2}B_n^3}\leqq\frac{|t|^3E\{\zeta^2_{nj}g(\zeta_{nj})\}}{ng(\sqrt{n})B_n^3}\leqq\frac{b_3|t|^3}{D_2^{3/2}ng(\sqrt{n})}.$$

Taking $\lambda=b_2/(4D_2)$, $\eta=b_2\sqrt{D_2}/(4b_3)$, we have for $|t|\leqq\eta g(\sqrt{n})$,

$$\left|\phi_{nj}\left(\frac{t}{\sqrt{n}B_n}\right)\right|\leqq1-\frac{\sigma^2_{nj}}{2nB_n^2}t^2+\frac{b_3|t|^3}{D_2^{3/2}ng(\sqrt{n})}$$

$$\leqq1-b_2t^2/(2D_2n)+b_2t^2/(4D_2n)=1-\lambda t^2/n\leqq\exp(-\lambda t^2/n)$$

and the lemma follows.

Lemma 4 Suppose that the conditions of lemma 1 hold. write

$$S_n=\sum_{j=1}^{n}\zeta_{nj}/(\sqrt{n}B_n)$$

whose c. f. is denoted by $f_n(t)$. Then there exists $\eta>0$ independent of n such that for n sufficiently large

$$\int_{|t|\leqq g(\sqrt{n})\eta}|t|^{-1}|f_n(t)-\exp(-t^2/2)|dt\leqq c/g(\sqrt{n}).$$

Proof Put

$$\tilde{S}_n=\sum_{j=1}^{n}\zeta_{nj}/(\sqrt{n}\tilde{B}_n),\qquad L_n=\sum_{j=1}^{n}E|\zeta_{nj}|^3/(\sum_{j=1}^{n}E\zeta^2_{nj})^{3/2}.$$

Denote c. f. of $\tilde{S}_n$ by $\tilde{f}_n(t)$, then by lemma 1 of [2], p. 109, we have

$$|\tilde{f}_n(t)-\exp(-t^2/2)|\leqq16L_n|t|^3\exp(-t^2/3),\ \text{for}\ |t|\leqq(4L_n)^{-1}.$$

Since $|\zeta_{nj}|\leqq\sqrt{n}$, $g(x)$ and $x/g(x)$ are non-decreasing for $x\geqq0$, it follows that $|\zeta_{nj}|\leqq\sqrt{n}g(|\zeta_{nj}|)/g(\sqrt{n})$. By (9), we have

$$L_n=\sum_{j=1}^{n}E|\zeta_{nj}|^3/(n^{3/2}\tilde{B}_n^3)\leqq\sum_{j=1}^{n}a_{nj}/[ng(\sqrt{n})(D_2^0)^{3/2}]\leqq D_1^0/[g(\sqrt{n})(D_2^0)^{3/2}].$$

Choosing $\eta_1=(D_2^0)^{3/2}/(4D_1^0)$, we have for $|t|\leqq\eta_1g(\sqrt{n})$

$$|\tilde{f}_n(t)-\exp(-t^2/2)|\leqq c|t|^3\exp(-t^2/3)/g(\sqrt{n}).$$

By (5) and (8), we have $|1-\tilde{B}_n/B_n|\leqq|1-\tilde{B}_n^2/B_n^2|\leqq c/g(\sqrt{n})\leqq1/4$ for n sufficiently large. Choose $\eta=4\eta_1/5$, then $|\tilde{B}_nt/B_n|\leqq\eta_1g(\sqrt{n})$ for $|t|\leqq\eta g(\sqrt{n})$. Since $|e^x-1|\leqq|x|e^{|x|}$, we have for $|t|\leqq\eta g(\sqrt{n})$:

$$\begin{aligned}|f_n(t)-\exp(-t^2/2)|&\leqq|\tilde{f}_n(\tilde{B}_nt/B_n)-\exp(-\tilde{B}_n^2t^2/2B_n^2)|\\&+|\exp(-\tilde{B}_n^2t^2/2B_n^2)-e^{-t^2/2}|\leqq c|\tilde{B}_nt/B_n|^3\exp(-\tilde{B}_n^2t^2/3B_n^2)/g(\sqrt{n})\\&+e^{-t^2/2}(t^2/2)|1-\tilde{B}_n^2/B_n^2|\exp(t^2\cdot|1-\tilde{B}_n^2/B_n^2|/2)\\&\leqq c(t^2+|t|^3)\exp(-t^2/4)/g(\sqrt{n})\end{aligned}$$

and the lemma is proved.

Lemma 5 Suppose that the conditions of lemma 1 hold, and

$$E\{e_j^4 g(e_j^2)\}\leqq c, \qquad j=1,\cdots,m,\ m\leqq n. \tag{12}$$

Further, let a_{nj}, $j=1,\cdots,n$, $n=1,2,\cdots$, be constants satisfiying $\sum_{j=1}^{n} a_{nj}^2\leqq 1$ for all n; Writing

$$e_{nj}^* = \hat{e}_{nj} - E(\hat{e}_{nj}),$$

then

$$E\Big(\sum_{j=1}^{m} a_{nj}e_{nj}^*\Big)^6 \leqq c\sqrt{n}/g(\sqrt{n}).$$

Proof. we have

$$E(\sum_{j=1}^{n} a_{nj}e_{nj}^*)^6\leqq \sum_{j=1}^{m} a_{nj}^6 E(e_{nj}^*)^6 + 15\sum_{j=1}^{m} a_{nj}^4 E(e_{nj}^*)^4 \sum_{j=1}^{m} a_{nj}^2 E(e_{nj}^*)^2$$

$$+20(\sum_{j=1}^{m} |a_{nj}|^3 E|e_{nj}^*|^3)^2 + 90(\sum_{j=1}^{m} a_{nj}^2 E(e_{nj}^*)^2)^3$$

$$\triangleq J_1+J_2+J_3+J_4.$$

Since $E|e_{nj}^*|^6\leqq 2^6 E|\hat{e}_{nj}|^6$, $\hat{e}_{nj}^2\leqq\sqrt{n}$, $\hat{e}_{nj}^2/g(\hat{e}_{nj}^2)\leqq\sqrt{n}/g(\sqrt{n})$, from (12) we have

$$J_1\leqq 2^6\sum_{j=1}^{m} a_{nj}^6 E(\hat{e}_{nj})^6 \leqq \frac{64\sqrt{n}}{g(\sqrt{n})}\sum_{j=1}^{m} a_{nj}^6 E\{\hat{e}_{nj}^4 g(\hat{e}_{nj}^2)\}$$

$$\leqq \frac{64\sqrt{n}}{g(\sqrt{n})}\sum_{j=1}^{m} a_{nj}^6 E\{e_j^4 g(e_j^2)\}\leqq 64C\sqrt{n}/g(\sqrt{n}).$$

Also, $\sum_{j=1}^{m} a_{nj}^2 E(e_{nj}^*)^2\leqq\sum_{j=1}^{m} a_{nj}^2 E(e_j^2)=\sigma^2\sum_{j=1}^{m} a_{nj}^2\leqq\sigma^2$, hence

$$J_2\leqq 15\sigma^2\sum_{j=1}^{m} a_{nj}^4 E(e_{nj}^*)^4\leqq 240\sigma^2\sum_{j=1}^{m} a_{nj}^4 E(\hat{e}_{nj})^4$$

$$\leqq 240\sigma^2\sum_{j=1}^{m} a_{nj}^4 [E\{\hat{e}_{nj}^4 I_{(\hat{e}_{nj}^2\leqq k)}\} + E\{\hat{e}_{nj}^4 g(\hat{e}_{nj}^2)\}/g(k)]$$

$$\leqq 240\sigma^2(k^2+c/g(k)),$$

where $k>0$ is chosen to make $g(k)>0$. Similar estimates hold for J_3, J_4, and the lemma is proved.

Lemma 6 Suppose that the conditions of lemma 1 hold, then

$$|\sqrt{n}B_n/\sqrt{\operatorname{var}\{(n-r)\hat{\sigma}_n^2\}}-1| = O(1/g(\sqrt{n})). \tag{13}$$

Proof Write $E(e_j^4)=\delta_j\sigma^4$, and

$$I_n - X'_n(X_nX'_n)^- X_n = M_n = (m_{nuv}), \qquad u,v=1,\cdots,n.$$

Then by the well-known formula

$$\text{Var}\{(n-r)\hat{\sigma}_n^2\} = \text{Var}\{e'_{(n)}M_n e_{(n)}\}$$

$$= \sigma^4\Big\{\sum_{j=1}^{n}(\delta_j-3)m^2_{njj} + 2\sum_{j=1}^{n}\sum_{k=1}^{n} m^2_{njk}\Big\}. \tag{14}$$

As M_n is symmetric and idempotent, one sees that

$$\sum_{j,k=1}^{n} m^2_{njk} = tr(M_nM'_n) = \text{rank}(M_n) = n-r. \tag{15}$$

Remembering (3), we have $m_{njj} = 1 - \sum_{u=1}^{r} a^2_{nuj}$, and

$$m^2_{njj} = 1 - 2\sum_{u=1}^{r} a^2_{nuj} + \Big(\sum_{u=1}^{r} a^2_{nuj}\Big)^2, \tag{16}$$

$$1 - 2\sum_{u=1}^{r} a^2_{nuj} \leqq m^2_{njj} \leqq 1 + r\sum_{u=1}^{r} a^2_{nuj}. \tag{17}$$

It follows from (14)—(17) that

$$\text{Var}\{(n-r)\hat{\sigma}_n^2\} = \sigma^4\Big\{\sum_{j=1}^{n}(\delta_j-1)m^2_{njj} - 2\sum_{j=1}^{n} m^2_{njj} + 2(n-r)\Big\}$$

$$\geqq \sigma^4\Big\{\sum_{j=1}^{n}(\delta_j-1) - 2\sum_{j=1}^{n}(\delta_j-1)\sum_{u=1}^{r} a^2_{nuj} - 2r\sum_{j=1}^{n}\sum_{u=1}^{r} a^2_{nuj} - 2n + 2(n-r)\Big\}.$$

Choose $k>0$ such that $g(\sqrt{k})>0$, write $w = g(\sqrt{k})/\sqrt{k}$ and we have for n sufficiently large

$$\delta_j\sigma^4 = E(e_j^4) = E\{e_j^4 I_{(e_j^2\leq\sqrt{n})}\} + E\{e_j^4 I_{(e_j^2>\sqrt{n})}\}$$

$$\leqq \sqrt{n}\,\sigma^2 + E\{e_j^4 g(e_j^2)\}/g(\sqrt{n}) \leqq \frac{n}{g(\sqrt{n})}(w\sigma^2 + D_1).$$

Also, $\sum_{j=1}^{n}(\delta_j-1)\sigma^4 = \sum_{j=1}^{n}\text{Var}(e_j^2) = nB_n^2$, and $\sum_{j=1}^{n}\sum_{u=1}^{r} a^2_{nuj} = r$,

we get

$$\text{Var}\{(n-r)\hat{\sigma}_n^2\} \geqq nB_n^2 - \frac{2rn}{g(\sqrt{n})}(w\sigma^2 + D_1) - 2r(r+1)\sigma^4. \tag{18}$$

A similar argument leads to

$$\text{Var}\{(n-r)\hat{\sigma}_n^2\} \leqq nB_n^2 + \frac{r^2n}{g(\sqrt{n})}(w\sigma^2 + D_1) + 2r\sigma^4. \tag{19}$$

Since $B_n^2 \geqq D_2 > 0$ and $g(\sqrt{n}) \to \infty$, (13) follows easily from (18) and (19).

Lemma 7 Let $w_n = w_{n1} + w_{n2}$, $n=1,2,\cdots$, be a sequence of ramdom variables; w_n, w_{n1} possess distribution functions H_n, H_{n1}, respectively; Further, suppose that $\|H_{n\cdot} - \Phi\| = O(1/g(\sqrt{n}))$ and there exists constant k such that $P(|w_{n2}| \geqq k/g(\sqrt{n})) = O(1/g(\sqrt{n}))$. Then

$$\|H_n-\Phi\|=O(1/g(\sqrt{n})). \tag{20}$$

Proof Evidently

$$H_{n1}(x-k/g(\sqrt{n}))-P(|w_{n2}|\geqq k/g(\sqrt{n}))\leqq H_n(x)$$
$$\leqq H_{n1}(x+k/g(\sqrt{n}))+P(|w_{n2}|\geqq k/g(\sqrt{n}))$$

and

$$|\Phi(x\pm k/g(\sqrt{n}))-\Phi(x)|\leqslant k/g(\sqrt{n}),$$

from which (20) follows easily.

Lemma 8 Let Z_n be a random variable with distribution $H_n(x)$. $w_n=a_nZ_n+b_n$, when $a_n>0$ and b_n are constants $(n=1,2,\cdots)$. Supposing that

$$|a_n-1|=O(1/g(\sqrt{n})),\quad b_n=O(1/g(\sqrt{n})),$$
$$\|H_n-\Phi\|=O(1/g(\sqrt{n})),$$

then

$$\|\widetilde{H}_n-\Phi\|=O(1/g(\sqrt{n})),$$

where $\widetilde{H}_n$ is the distribution of w_n.

The proof is easy and therefore omitted.

3. Proof of The Theorem

By lemmas 1 and 2, there exist constants $L>0$, $b_j>0$, $j=1,2,3$, such that

$$\#(A_n)\geqq b_1n,\quad \#(A_n\cap H_n)\geqq b_1n/2,$$

where A_n and H_n were defined earlier. Putting

$$\Pi_n'=\{v:\ 1\leqq v\leqq n;\ E(e_v^4g(e_v^2))\geqq 4D_1/b_1\},$$

then $\#(\Pi_n')\leqq b_1n/4$, hence

$$\#(A_n\cap H_n-\Pi_n')\geqq b_1n/4.$$

Pick arbitrarily $\mu_n=\min\,(n^{4/3}/g(\sqrt{n}),\ b_1n/4)$ elements from the set $A_n\cap H_n\cdot\Pi_n'$ to form a set Π_n'', and define $\Pi_n=\{1,2,\cdots,n\}-\Pi_n'-\Pi_n''$. Obviously

$$\#(\Pi_n\cap A_n)\geqq b_1n/4.$$

Define

$$S_n=\sum_{j=1}^{n}\zeta_{nj}/(\sqrt{n}B_n),$$

$$\Delta_n=8\sum_{u=1}^{r}\Big(\sum_{v\in\Pi_n}a_{nuv}e_{nv}^*\Big)^2/(\sqrt{n}B_n),$$

$$\Delta_n'=8\sum_{u=1}^{r}\Big(\sum_{v\in\Pi_n'}a_{nuv}e_{nv}^*\Big)^2/(\sqrt{n}B_n),$$

$$\Delta_n''=8\sum_{u=1}^{r}\Big(\sum_{v\in\Pi_n''}a_{nuv}e_{nv}^*\Big)^2/(\sqrt{n}B_n).$$

$$Q_n^{(1)}=S_n-2\sum_{u=1}^{r}\Big(\sum_{v=1}^{n}a_{nuv}e_{nv}^*\Big)^2/(\sqrt{n}B_n),$$

$$T_n=S_n-\Delta_n-\Delta_n'.$$

Then

$$T_n-\Delta_n''\leqq Q_n^{(1)}\leqq S_n.$$

Now we proceed to show that

$$\sup_x|P(Q_n^{(1)}\leqq x)-\Phi(x)|=O(1/g(\sqrt{n})). \tag{21}$$

Denote the c. f; of S_n and T_n by $f_n(t)$ and $\psi_n(t)$, respectively; Choose $\eta>0$ such that the conclusion of lemmas 3 and 4 are true; Then we have for n large,

$$\int_{|t|\leq \eta g(\sqrt{n})}\frac{1}{|t|}|f_n(t)-e^{-t^2/2}|dt=O(1/g(\sqrt{n})) \tag{22}$$

and by Berry-Esseen,

$$\sup_x|P(S_n\leqq x)-\Phi(x)|=O(1/g(\sqrt{n})). \tag{23}$$

On the other hand, by Markov and Marcinkiewicz inequality, we have

$$P(|\Delta_n''|\geqq g^{-1}(\sqrt{n}))\leqq Cg^3(\sqrt{n})n^{-3/2}\sum_{u=1}^{r}E\,(\sum_{v\in\Pi_n''}a_{nuv}e_{nv}^*)^6$$

$$\leqq Cg^3(\sqrt{n})n^{-3/2}\sum_{u=1}^{r}E\,(\sum_{v\in\Pi_n''}a_{nuv}^2e_{nv}^{*2})^3$$

$$\leqq Cg^3(\sqrt{n})n^{-3/2}n^{-3}E\,(\sum_{v\in\Pi_n''}e_{nv}^{*2})^3$$

$$\leqq Cg^3(\sqrt{n})n^{-9/2}\mu_n^2\sum_{v\in\Pi_n''}E(\hat{e}_{nv}^6)$$

$$\leqq Cg^3(\sqrt{n})n^{-9/2}\mu_n^2(\sqrt{n}/g(\sqrt{n}))\sum_{v\in\Pi_n''}E\{\hat{e}_{nv}^4g(\hat{e}_{nv}^2)\}$$

$$\leqq Cg^2(\sqrt{n})n^{-4}\mu_n^3.$$

If $\mu_n=n^{4/3}/g(\sqrt{n})$, then

$$P(|\Delta_n''|\geqq g^{-1}(\sqrt{n}))\leqq c/g(\sqrt{n}),$$

If $\mu_n=b_1n/4$, then $g(\sqrt{n})\leqq 4n^{1/3}/b_1$, and

$$P(|\Delta_n''|\geqq g^{-1}(\sqrt{n}))\leqq Cg^2(\sqrt{n})/n\leqq C/g(\sqrt{n}).$$

Hence we have $P(|\Delta_n''|\geqq g^{-1}(\sqrt{n})=O(1/g(\sqrt{n}))$. By lemma 7, the truth of (21) follows from

$$\sup_x|P(T_n\leqq x)-\Phi(x)|=O(1/g(\sqrt{n})), \tag{24}$$

which we now proceed to prove. According to Berry-Esseen, we need to show that for n large

$$\int_{|t|\leq \eta g(\sqrt{n})} \frac{1}{|t|}|\psi_n(t)-e^{-t^2/2}|\,dt = O(1/g(\sqrt{n})). \tag{25}$$

We have

$$\begin{aligned}
&\int_{|t|\leq \eta g(\sqrt{n})} \frac{1}{|t|}|\psi_n(t)-e^{-t^2/2}|\,dt \\
&\leqq \int_{|t|\leq \eta g(\sqrt{n})} \frac{1}{|t|}|f_n(t)-e^{-t^2/2}|\,dt \\
&+\int_{|t|\leq \eta g(\sqrt{n})} \frac{1}{|t|}|E\{e^{itS_n}(1-e^{-it\Delta_n'})\}|\,dt \\
&+\int_{|t|\leq \eta g(\sqrt{n})} \frac{1}{|t|}|E\{e^{it(S_n-\Delta_n')}(1-e^{-it\Delta_n})\}|\,dt = J_1+J_2+J_3.
\end{aligned} \tag{26}$$

From (22)

$$J_1 = O(1/g(\sqrt{n})). \tag{27}$$

Putting $h=b_1\lambda/4$, using lemma 3 and observing that $\Pi_n'\cap\Pi_n''=\emptyset$ and $\Pi_n'\cap\Pi_n=\emptyset$, we have for n large

$$\begin{aligned}
J_2 &\leqq \int_{|t|\leq \eta g(\sqrt{n})} \frac{1}{|t|}|E\{\exp(it\sum_{j\in\Pi_n\cap A_n}\zeta_{nj}/\sqrt{n}B_n)\}|E|1-e^{it\Delta_n'}|\,dt \\
&\leqq \int_{|t|\leq \eta g(\sqrt{n})} \{\exp(-\lambda t^2/n)\}^{b_1 n/4}\cdot E|\Delta_n'|\,dt \\
&\leqq \frac{C}{\sqrt{n}}\int_{|t|\leq \eta g(\sqrt{n})} e^{-ht^2}E\Big\{\sum_{u=1}^{r}\Big(\sum_{v\in\Pi_n'} a_{nuv}e^*_{nv}\Big)^2\Big\}dt \\
&\leqq \frac{c}{g(\sqrt{n})}\int_0^\infty e^{-ht^2}dt = O(1/g(\sqrt{n})).
\end{aligned} \tag{28}$$

For estimating J_3, consider two cases:

1. $\mu_n=b_1n/4$. In this case $g(\sqrt{n})\leqq 4n^{1/3}/b_1$, and

$$\begin{aligned}
J_3 &\leqq \int_{|t|\leq \eta g(\sqrt{n})} |E\{\exp(it\sum_{j\in\Pi_n''}\zeta_{nj}/\sqrt{n}B_n)\}|E|\Delta_n|\,dt \\
&\leqq \frac{C}{\sqrt{n}}\int_{|t|\leq \eta g(\sqrt{n})} (\exp(-\lambda t^2/n))^{b_1 n/4}E\Big\{\sum_{u=1}^{r}\Big(\sum_{v\in\Pi_n} a_{nuv}e^*_{nv}\Big)^2\Big\}dt \\
&\leqq \frac{c}{\sqrt{n}}\int_0^\infty e^{-ht^2}dt = O(1/\sqrt{n}) = O(1/g(\sqrt{n})).
\end{aligned} \tag{29}$$

2. $\mu_n=n^{4/3}/g(\sqrt{n})$. In this case $g(\sqrt{n})>4n^{1/3}/b_1$. Noticing that $\Pi_n\cap\Pi_n'=\emptyset$, by lemma 5 we have

$$E(\Delta_n^3) = O(n^{-1}g^{-1}(\sqrt{n})).$$

Hence

$$E(\Delta_n^2)\leqq(E\Delta_n^3)^{2/3} = O(n^{-2/3}g^{-2/3}(\sqrt{n})). \tag{30}$$

Now note that

$$J_3 \leqq \int_{|t|\leqq \eta g(\sqrt{n})} |E\{e^{it(s_n-\Delta_n')}\Delta_n\}|\,dt + \int_{|t|\leqq \eta g(\sqrt{n})} |t|\,|E\{e^{it(s_n-\Delta_n')}\theta_n\Delta_n^2\}|\,dt = J_4 + J_5. \tag{31}$$

Here $|\theta_n|\leqq 1$, and θ_n depends only on $\{e_{nv}^* : v\in \Pi_n\}$. Therefore θ_n, $\{e_v : v\in \Pi_n'\}$ and $\{\zeta_{nv} : v\in \Pi_n''\}$ are mutually independent; Hence

$$|E\{e^{it(s_n-\Delta_n')}\theta_n\Delta_n^2\}| \leqq E(\Delta_n^2)\,|E\{\exp(it\sum_{v\in \Pi_n''}\zeta_{nv}/\sqrt{n}B_n)\}|$$
$$\leqq Cn^{-2/3}g^{-2/3}(\sqrt{n})\{\exp(-\lambda t^2/n)\}^{n^{4/5}/g(\sqrt{n})}$$
$$\leqq Cn^{-2/3}g^{-2/3}(\sqrt{n})\exp(-\lambda n^{1/3}t^2 g^{-1}(\sqrt{n})),$$

So, remembering that $g(x)=O(x)$ as $x\to\infty$, we have

$$\int_{g^{1/2}(\sqrt{n})\leqq|t|\leqq \eta g(\sqrt{n})} |t|\cdot|E\{e^{it}\theta^{(s_n-\Delta_n')}{}_n\Delta_n^2\}|\,dt$$
$$\leqq Cn^{-2/3}g^{-2/3}(\sqrt{n})\exp(-\lambda n^{1/3})\int_{|t|\leqq \eta g(\sqrt{n})} |t|\,dt$$
$$\leqq Cn^{-2/3}g^{4/3}(\sqrt{n})\exp(-\lambda n^{1/3}) = O(g^{-1}(\sqrt{n})). \tag{32}$$

Using again $g(x)=O(x)$,

$$\int_{|t|\leqq g^{1/2}(\sqrt{n})} |t|\cdot|E\{e^{it(s_n-\Delta_n')}\theta_n\Delta_n^2\}|\,dt$$
$$\leqq \int_{|s|\leqq g^{1/2}(\sqrt{n})} |t|\,E(\Delta_n^2)\,dt \leqq Cn^{-2/3}g^{-2/3}(\sqrt{n})g(\sqrt{n})$$
$$= O(g^{-1}(\sqrt{n})). \tag{33}$$

By (31)—(33), we get

$$J_5 = O(g^{-1}(\sqrt{n})). \tag{34}$$

Further,

$$|E\{e^{it(s_n-\Delta_n')}\cdot\Delta_n\}|$$
$$\leqq \frac{C}{\sqrt{n}}\sum_{u=1}^{r}\sum_{v\in\Pi_n} a_{nuv}^2\,|E\{e_{nuv}^{*2}\exp(it(S_n-\Delta_n'))\}|$$
$$+\frac{C}{\sqrt{n}}\sum_{u=1}^{r}\sum_{\substack{v\neq v'\\ v,v'\in\Pi_n}} |a_{nuv}a_{nuv'}|\cdot|E\{e_{nv}^{*}e_{nv'}^{*}\exp(it(S_n-\Delta_n'))\}|$$
$$= M_1 + M_2, \tag{35}$$

where

$$M_1 \leqq \frac{C}{\sqrt{n}}\sum_{u=1}^{r}\sum_{v\in\Pi_n} a_{nuv}^2\,|E\{\exp(\,it\sum_{j\in\Pi_n\cap A_n, j\neq v}\zeta_{nj}/\sqrt{n}B_n)\}|$$
$$\times|E\{\exp(it\zeta_{uv}/\sqrt{n}B_n)e_{nv}^{*2}\}|$$

$$\leqq \frac{C}{\sqrt{n}} \sum_{u=1}^{r} \sum_{v=1}^{n} a_{nuv}^2 \{\exp(-\lambda t^2/n)\}^{b_1 n/4} \cdot \sigma^2 \tag{36}$$

$$\leqq \frac{c}{\sqrt{n}} \exp(-ht^2).$$

Likewise,

$$M_2 \leqq \frac{C}{\sqrt{n}} \exp(-ht^2) \sum_{u=1}^{r} \Big(\sum_{v=1}^{n} |a_{nuv}| q_{uv} \Big)^2, \tag{37}$$

where $q_{nv} = |E\{e_{nv}^* \cdot \exp(it\zeta_{nv}/\sqrt{n} B_n)\}|$.

Since

$$q_{nv}^2 \leqq (tE|e_{nv}^* \zeta_{nv}|)^2/(nB_n^2) \leqq Cn^{-1} t^2 E(e_{nv}^{*2}) E(\zeta_{nv}^2) \leqq Cn^{-1} \sigma^2 d_v^2 t^2 \tag{38}$$

(for the meaning of d_v, see (5)) by (37), (38),

$$M_2 \leqq \frac{C}{\sqrt{n}} \exp(-ht^2) \sum_{u=1}^{r} \sum_{v=1}^{n} a_{nuv}^2 \sum_{v=1}^{n} q_{nv}^2 \leqq \frac{C}{\sqrt{n}} \exp(-ht^2) t^2 \frac{1}{n} \sum_{v=1}^{n} d_v^2.$$

Since $\sum_{v=1}^{n} d_v^2 = O(n)$ (which follows easily from (4)), we get

$$M_2 \leqq \frac{c}{\sqrt{n}} t^2 \exp(-ht^2). \tag{39}$$

From (31), (35), (36), (39), we get

$$J_4 \leqq \frac{c}{\sqrt{n}} \int_{|t| \leq \eta g(\sqrt{n})} (1+t^2) \exp(-ht^2) dt$$

$$\leqq C/\sqrt{n} = O(g^{-1}(\sqrt{n})). \tag{40}$$

By (31), (34) and (40), we obtain again for the present case

$$J_3 = O(g^{-1}(\sqrt{n})). \tag{41}$$

From (26)—(29) and (41) we finally get (25), which in turn proves (24) and eventually (21), as stated earlier.

Now define

$$Q_n^{(2)} = \frac{1}{\sqrt{n} B_n} \Big(\sum_{j=1}^{n} \hat{e}_{nj}^2 - (n-r)\sigma^2 \Big) - \frac{1}{\sqrt{n} B_n} \sum_{u=1}^{r} \Big(\sum_{v=1}^{n} a_{nuv} \hat{e}_{uv} \Big)^2. \tag{42}$$

We have

$$\Big| \Big(\sum_{j=1}^{n} \hat{e}_{nj}^2 - (n-r)\sigma^2 \Big) \Big/ \sqrt{n} B_n - S_n \Big|$$

$$\leqq \Big(\sum_{j=1}^{n} (\sigma^2 - E(\hat{e}_{nj}^2)) + r\sigma^2 \Big) \Big/ \sqrt{n} B_n$$

$$= \Big(\sum_{j=1}^{n} \int_{|x| > n^{1/4}} x^2 dF_j + r\sigma^2 \Big) \Big/ \sqrt{n} B_n$$

$$\leqq \frac{1}{\sqrt{n}B_n}\left(\frac{1}{\sqrt{n}\,g(\sqrt{n})}\sum_{j=1}^{n}E\{e_j^4g(e_j^2)\}+r\sigma^2\right)$$

$$\leqq \frac{1}{\sqrt{n}D_2}\left(\frac{D_1\sqrt{n}}{g(\sqrt{n})}+r\sigma^2\right)=O(g^{-1}(\sqrt{n})). \tag{43}$$

By Cauchy-Schwarz,

$$|E(\hat{e}_{nj})|^2=\left(\int_{|x|>n^{1/4}}xdF_j\right)^2\leqq\int_{|x|>n^{1/4}}x^2dF_j\int_{|x|>n^{1/4}}dF_j$$

$$\leqq\int_{|x|>n^{1/4}}x^2dF_j\leqq\frac{1}{\sqrt{n}\,g(\sqrt{n})}E\{e_j^4g(e_j^2)\}.$$

Hence

$$\left(\sum_{v=1}^{n}a_{nuv}\hat{e}_{uv}\right)^2=\left(\sum_{v=1}^{n}a_{nuv}e_{nv}^*+\sum_{v=1}^{n}E(\hat{e}_{nv})\right)^2$$

$$\leqq 2\left(\sum_{v=1}^{n}a_{nuv}e_{nv}^*\right)^2+2\left(\sum_{v=1}^{n}a_{nuv}E(\hat{e}_{uv})\right)^2$$

$$\leqq 2\left(\sum_{v=1}^{n}a_{nuv}e_{nv}^*\right)^2+2\sum_{v=1}^{n}E^2(\hat{e}_{nv})\leqq 2\left(\sum_{v=1}^{n}a_{nuv}e_{nv}^*\right)^2+2D_1\sqrt{n}/g(\sqrt{n}). \tag{44}$$

Form (42)—(44), and $B_n\geqq\sqrt{D_2}>0$, we see

$$Q_n^{(1)}-\frac{C}{g(\sqrt{n})}\leqq Q_n^{(2)}\leqq S_n+\frac{C}{g(\sqrt{n})}. \tag{45}$$

By (21), (23), (45) and employing lemma 7, we obtain

$$\operatorname*{snp}_x|P(Q_n^{(2)}\leqq x)-\Phi(x)|=O(g^{-1}(\sqrt{n})). \tag{46}$$

Let

$$Q_n^{(3)}=(n-r)(\hat{\sigma}_n^2-\sigma^2)/\sqrt{n}B_n$$

$$=\frac{1}{\sqrt{n}B_n}\left(\sum_{j=1}^{n}e_j^2-(n-r)\sigma^2\right)-\frac{1}{\sqrt{n}B_n}\sum_{u=1}^{r}\left(\sum_{v=1}^{r}a_{nuv}e_v\right)^2. \tag{47}$$

Comparing (47) with (42), one finds that the diffrence between $Q_n^{(2)}$ and $Q_n^{(3)}$ lies only in the fact that $\hat{e}_{nj}$ in $Q_n^{(2)}$ is replaced by e_j in $Q_n^{(3)}$. Hence for every x,

$$|P(Q_n^{(3)}\leqq x)-P(Q_n^{(2)}\leqq x)|\leqq\sum_{j=1}^{n}P(e_j\neq\hat{e}_{nj})$$

$$\leqq\sum_{j=1}^{n}\int_{|x|>n^{1/4}}dF_j\leqq\frac{1}{ng(\sqrt{n})}\sum_{j=1}^{n}E\{e_j^4g(e_j^2)\}=O(g^{-1}(\sqrt{n})).$$

From this and (46) we obtain

$$\sup_x|P(Q_n^{(3)}\leqq x)-\Phi(x)|=O(g^{-1}(\sqrt{n})). \tag{48}$$

Finally, we have

$$Z_n = (\hat{\sigma}_n^2 - \sigma^2)/\sqrt{\operatorname{var}(\hat{\sigma}_n^2)} = \frac{\sqrt{n}\,B_n}{(n-r)\sqrt{\operatorname{var}(\hat{\sigma}_n^2)}} Q_n^{(3)}. \tag{49}$$

By (13), (48), (49), and employing lemma 8, we reach (6). This concludes the proof of the theorem.

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