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Characteristic Semi-simple Lie Algebras and Completely Semi-simple Lie Algebras over any Field*

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In this paper, we study some properties of the finite dimensional characteristic semi-simple ($C.S.S.$) Lie algebras and completely semi-simple Lie algebras over any field. The definitions and some results of these algebras have been given by G. B. Seligman in [1].

We can easily prove the following lemmas:

Lemma 1 The centre of any finite dimensional Lie algebra L is a characteristic ideal of L.

Lemma 2 The centre of characteristic semi-simple Lie algebra is zero.

Lemma 3 If D is the derivation of Lie algebra L, then $[\mathrm{ad}x, D]=\mathrm{ad}(xD)$, for any $x\in L$. (See[2]).

Theorem 1 Let L be a finite dimensional Lie algebra over any field F, the centre of L be $\{0\}$, and $\mathscr{D}(L)$ be the derivation algebra of L. Then $\mathscr{D}(L)$ is semi-simple, if and only if L is C. S. S.

Proof "$\rightarrow$": Let R be the characteristic radical of L, for $\forall x\in R$, $\forall D\in\mathscr{D}(L)$, we have $xD\in R$ and $[\mathrm{ad}x, D]=\mathrm{ad}(xD)$ (lemma3), but $\mathrm{ad}(xD)\in \mathrm{ad}R$, so $\mathrm{ad}R$ is the solvable ideal of $\mathscr{D}(L)$; Since $\mathscr{D}(L)$ is semi-simple, it follows that $\mathrm{ad}R=0$ and R is in the centre of L; Hence $R=0$ and L is C. S. S.

"$\leftarrow$": ad L is the ideal of $\mathscr{D}(L)$. $L\cong_{\varphi}\mathrm{ad}L$, since the centre of L is 0. Now assume that $R_{\mathscr{D}(L)}$ is the radical of $\mathscr{D}(L)$. Let $R=R_{D(L)}\cap\mathrm{ad}\,L$, R is solvable ideal of $\mathscr{D}(L)$ and $\mathrm{ad}L$; Put $R=\mathrm{ad}\,L_1$, then L_1 is the solvable ideal of L. For any $D\in\mathscr{D}(L)$, any $x\in L_1$, according to lemma 3, $\mathrm{ad}(xD)=[\mathrm{ad}x, D]\in R$, thus $xD\in L_1$ and $L_1D\subseteq L_1$ and therefore L_1 is the solvable characteristic ideal of L. But L is C. S. S., then $L_1=\{0\}$; Hence $R=\mathrm{ad}\,L_1=0$, it follows that

$$[R_{\mathscr{D}(L)}, \mathrm{ad}L]=0, \tag{1}$$

For $\forall a, b\in L$, $\forall D\in R_{\mathscr{D}(L)}$, we have $[a,bD]=[a, b]D-[aD,b]=a(\mathrm{ad}\ b)D-aD(\mathrm{ad}\,b)=a[\mathrm{ad}b,D]=0$ (according to(1)); bD belongs to the centre of L, since a is any element of L; Therefore $bD=0$; but b is any element of L, then $D=0$ and $R_{\mathscr{D}(L)}=0$, hence we find that $\mathscr{D}(L)$ is semi-simple.

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Jacobson has given an example in [2] to show that in the case of characteristic $p \neq 0$, there is semi-simple Lie algebra L, which is not the direct sum of simple ideals. Furthermore, if L is C. S. S. over the field of characteristic $p \neq 0$, but not semi-simple, according to Theorem 1, we know that the derivation algebra $\mathscr{D}(L)$ is semi-simple. But its ideal ad$L \cong L$ is not semi-simple. These facts show that the principal structure theorem of semi-simple Lie algebra, which has been obtained in the case of characteristic o, cannot be generalized to any field. But we have the following two theorems:

Theorem 2 Let L be a semi-simple Lie algebra over any field. Then L is the direct sum of simple ideals, if and only if L is also completely semi-simple.

Proof The necessity is evident. We now prove the sufficiency as follows. If L is completely semi-simple, according to the definition (See[1]), L has the decomposition: $L = L_1 \oplus \cdots \oplus L_s$, where L_i, $i = 1, 2, \cdots, s$ are ideals of L, and characteristic simple. If there exists L_i which is not simple, then L_i has proper ideal $R \neq 0$. Seligman has proved in [1] that any proper ideal of the characteristic simple Lie algebra is nilpotent ideal, so R is the nilpotent ideal of L_i. It is easy to see that R is also the nilpotent ideal of L, which is in contradiction with semi-simplicity of L. Thus all of L_i are simple ideals of L.

Theorem 3 If semi-simple Lie algebra L is completely semi-simple, then any nonzero ideal of L is semi-simple.

Proof We know from theorem 2 that L is the direct sum of the simple ideals: $L = L_1 \oplus \cdots \oplus L_s$; $L_i \neq 0$, $[L_i, L_i] = L_i$. Let A be any nonzero ideal of L. It is easy to see that there is one L_i at least so that $A \cap L_i \neq \{0\}$, and so $L_i \subseteq A$. Now suppose $L_{i_1}, \cdots, L_{i_k}$ are contained in A, and the rest $L_{i_{k+1}}, L_{i_{k+2}}, \cdots, L_{i_s}$ are not contained in A. Then

$$L_{i_1} \oplus \cdots \oplus L_{i_k} \subseteq A, \tag{2}$$

for any $L_{i_{k+j}}$, $j = 1, 2, \cdots, s-k$ we have $A \cap L_{i_{k+j}} = 0$, hence $[A. L_{i_{k+j}}] \subseteq A \cap L_{i_{k+j}} = 0$ and therefore

$$[A, L_{i_{k+1}} \oplus \cdots \oplus L_{i_s}] = 0, \tag{3}$$

for $x \in A \cap (L_{i_{k+1}} \oplus \cdots \oplus L_{i_s})$, using (2) and (3), gives $x = 0$, thus $A \cap (L_{i_{k+1}} \oplus \cdots \oplus L_{i_s}) = 0$. This shows that $A \oplus (L_{i_{k+1}} \oplus \cdots \oplus L_{i_s})$ is the direct sum of linear spaces: $L = (L_{i_1} \oplus \cdots \oplus L_{i_k}) \oplus (L_{i_{k+1}} \oplus \cdots \oplus L_{i_s}) \subseteq A \oplus (L_{i_{k+1}} \oplus \cdots \oplus L_{i_s}) \subseteq L$. Using again (2) and the relation of the dimensions, we can obtain $A = L_{i_1} \oplus \cdots \oplus L_{i_k}$. Now assume that R is any solvable ideal of A, thus $[R, L] = [R, A \oplus (L_{i_{k+1}} \oplus \cdots \oplus L_{i_s})] = [R, A] \subseteq R$, Hence R is the solvable ideal of L. But L is semi-simple, it follows that $R = 0$ and A is semi-simple.

References

[1] Seligman, G. B., Characteristic ideals and the structure of Lie algebras, Proc. Amer. Math. soc, 8 (1957), 159-164.

[2] Jacobson, N., Lie Algebras, Intercience, New York, 1962.