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On Flat and Coflat Modules*

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It is well known that a ring R is left hereditary iff every left ideal of R is projective, iff every submodule of a projective module is projective (ld $R\leqslant 1$), iff every quotient module of an injective module is injective (lcd $R\leqslant 1$), where ld R and lcd R means the left global dimension resp. codimension of the ring R. These rings may be generalized to those of weak left global dimension at most 1 (wld $R \leqslant 1$). The latter condition holds iff every left ideal of R is flat, iff every submodule of a flat module is flat and is a generalization of the left semi-hereditity as well. ([3], ch. 4 and 9.) We introduce the notion of coflatness to characterize a ring for which every quotient module of a coflat module is coflat. It turns out that the class of rings rising thereby is exactly that of the left semi-hereditary ones.

All rings considered are associative with an identity element and all R-(right) modules are unitary. ${}_R\underline{M}$ ($\underline{M}_R$) means the category of R-(right) modules.

Proposition 1 (homotopy lemma) Given ${}_R\underline{M}$- diagram with exact sequences and commutative squares. Then f'' is liftable iff there exist g' : $X\to Y'$ and g'': $X''\to Y$ such that $f=v'g'+g''u$ iff f' is extendable.

$$\begin{tikzcd} & X' \arrow[r, "u'"] \arrow[d, "f'"'] & X \arrow[r, "u"] \arrow[d, "f"] \arrow[dl, dashed, "g'"'] & X'' \arrow[r] \arrow[d, "f''"] \arrow[dl, dashed, "g''"'] & O \\ O \arrow[r] & Y' \arrow[r, "v'"] & Y \arrow[r, "v"] & Y'' & \end{tikzcd}$$

An exact ${}_R\underline{M}$- sequence $O\to A\to B\to C\to O$ is known to be pure exact iff for each right module X the sequence $O\to X\otimes A\to X\otimes B\to X\otimes C\to O$ is exact. A criterion hereof is that for each finitely presented module E the sequence $O\to \mathrm{Hom}_R(E, A)\to \mathrm{Hom}_R(E, B)\to \mathrm{Hom}_R(E, C)\to O$ is exact. The equivalence is a consequence of the homotopy lemma and a result of Cohn ([1], Thm 2.4). In view of this criterion a module Q is flat iff each exact sequence $O\to A\to B\to Q\to O$ is pure exact.

Proposition 2 Let P, I be R-modules.

(1) If $K\leqslant P'\leqslant P$ and K is pure in P, then K is pure in P';

(2) If I'' is a quotient module of I, Q is a quotient module of I'' and $I\to Q\to O$ is pure exact, then $I''\to Q\to O$ is pure exact;

Proposition 3 Let $O\to Q'\to Q\to Q''\to O$ be exact ${}_R\underline{M}$- sequence; If Q' and Q'' are flat, then Q it flat.

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Proof Use the purity criterion and a result of Lazard ([2], cor. 1. 3).

Proposition 4 Let R be a ring. R is left semi-hereditary iff R is left coherent and every left ideal of R is flat.

Definition An R-module I is said to be coflat iff each exact sequence $O\to I\to B\to C\to O$ is pure exact.

Proposition 5 An R-module I is coflat iff for every finitely generated R-module N and every R-homomorphism f: $N\to$I f can be extended to R^n, $n\in \mathbb{N}$.

Proof (Necessity only.) Using the homotopy lemma it is sufficient to show that h: $R^n/N\to Q/I$ is liftable. But R^n/N is finitely presented and I is coflat.

Proposition 6 Let $O\to I'\to I\to I''\to O$ be exact ${}_R\underline{M}$- sequence. If I' and I'' are coflat, then I is coflat.

Proof Check for I the condition given in prop. 5.

Proposition 7 Let R be a ring. The following conditions are equivalent:

(a) every quotient module of an injective module is coflat,

(b) every quotient module of a coflat module is coflat,

(c) R is left semi-hereditary.

Proof (a)$\Rightarrow$(b). Suppose I is a coflat module and K its arbitrary submodule. To show that I/K is coflat, choose any injective embedding $O\to I\to E$. Then E,E/K and E/I are interrelated just like I, I'' and Q in prop. 2(2). So E/I is a pure quotient module of E/K. Use homotopy lemma to check for I/K being coflat.

(b)$\Rightarrow$(c). Let N be finitely generated left ideal of R. We need to show that for each R-module X the exact sequence $X\xrightarrow{v}N\to O$ splits. Choose any injective embedding $O\to X\xrightarrow{u}I$. Then $O\to X'\to I\to I/X'\to O$ is exact with $X'=\ker$ u. By [3], lemma 4. 22, it is sufficient to show that any f: $N\to I/X'$ is liftable. f extends to g: $R^n\to I/X'$ by the coflatness of I/X'. g lifts to h: $R^n\to I$ since R^n is projective. It is easy to see that the restriction of h on N has X as its range. Thus there exists a k: $N\to X$ which splits v.

(c)$\Rightarrow$(a). Suppose I'' is a quotient module of an injective module I. To check for I'' the condition given in prop. 5 is straightforward.

If we introduce the notion of weak left global codimension (wlcd) for a ring via coflat resolution in an obvious way, then rings satisfying the equivalent conditions in prop. 7 are exactly those having wlcd$\leqslant$1. The dual couple of wld -wlcd is therefore no longer symmetrical as its precedent ld - lcd.

References

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