

On Flat and Coflat Modules*

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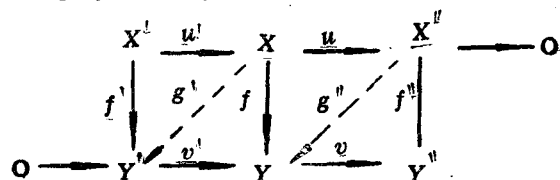
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It is well known that a ring R is left hereditary iff every left ideal of R is projective, iff every submodule of a projective module is projective ($\text{ld } R \leq 1$), iff every quotient module of an injective module is injective ($\text{lcd } R \leq 1$), where $\text{ld } R$ and $\text{lcd } R$ means the left global dimension resp. codimension of the ring R . These rings may be generalized to those of weak left global dimension at most 1 ($\text{wld } R \leq 1$). The latter condition holds iff every left ideal of R is flat, iff every submodule of a flat module is flat and is a generalization of the left semi-heredity as well. ([3], ch. 4 and 9.) We introduce the notion of coflatness to characterize a ring for which every quotient module of a coflat module is coflat. It turns out that the class of rings rising thereby is exactly that of the left semi-hereditary ones.

All rings considered are associative with an identity element and all R -(right) modules are unitary. ${}_R\mathbf{M}(\mathbf{M}_R)$ means the category of R -(right) modules.

Proposition 1 (homotopy lemma)

Given ${}_R\mathbf{M}$ -diagram with exact sequences and commutative squares. Then f'' is liftable iff there exist $g' : X \rightarrow Y'$ and $g'' : X'' \rightarrow Y$ such that $f = v'g' + g''u$ iff f' is extendable.



An exact ${}_R\mathbf{M}$ -sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is known to be pure exact iff for each right module X the sequence $0 \rightarrow X \otimes A \rightarrow X \otimes B \rightarrow X \otimes C \rightarrow 0$ is exact. A criterion hereof is that for each finitely presented module E the sequence $0 \rightarrow \text{Hom}_R(E, A) \rightarrow \text{Hom}_R(E, B) \rightarrow \text{Hom}_R(E, C) \rightarrow 0$ is exact. The equivalence is a consequence of the homotopy lemma and a result of Cohn ([1], Thm 2.4). In view of this criterion a module Q is flat iff each exact sequence $0 \rightarrow A \rightarrow B \rightarrow Q \rightarrow 0$ is pure exact.

Proposition 2 Let P, I be R -modules.

- (1) If $K \leq P' \leq P$ and K is pure in P , then K is pure in P' ;
- (2) If I'' is a quotient module of I , Q is a quotient module of I'' and $I \rightarrow Q \rightarrow 0$ is pure exact, then $I'' \rightarrow Q \rightarrow 0$ is pure exact;

Proposition 3 Let $0 \rightarrow Q' \rightarrow Q \rightarrow Q'' \rightarrow 0$ be exact ${}_R\mathbf{M}$ -sequence; If Q' and Q'' are flat, then Q is flat.

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Proof Use the purity criterion and a result of Lazard ([2], cor. 1. 3).

Proposition 4 Let R be a ring. R is left semi-hereditary iff R is left coherent and every left ideal of R is flat.

Definition An R -module I is said to be coflat iff each exact sequence $0 \rightarrow I \rightarrow B \rightarrow C \rightarrow 0$ is pure exact.

Proposition 5 An R -module I is coflat iff for every finitely generated R -module N and every R -homomorphism $f: N \rightarrow I$ f can be extended to R^n , $n \in \mathbb{N}$.

Proof (Necessity only.) Using the homotopy lemma it is sufficient to show that $h: R^n/N \rightarrow Q/I$ is liftable. But R^n/N is finitely presented and I is coflat.

Proposition 6 Let $0 \rightarrow I' \rightarrow I \rightarrow I'' \rightarrow 0$ be exact ${}_R\mathbf{M}$ -sequence. If I' and I'' are coflat, then I is coflat.

Proof Check for I the condition given in prop. 5.

Proposition 7 Let R be a ring. The following conditions are equivalent:

- (a) every quotient module of an injective module is coflat,
- (b) every quotient module of a coflat module is coflat,
- (c) R is left semi-hereditary.

Proof (a) \Rightarrow (b). Suppose I is a coflat module and K its arbitrary submodule. To show that I/K is coflat, choose any injective embedding $0 \rightarrow I \rightarrow E$. Then $E/E/K$ and E/I are interrelated just like I , I'' and Q in prop. 2(2). So E/I is a pure quotient module of E/K . Use homotopy lemma to check for I/K being coflat.

(b) \Rightarrow (c). Let N be finitely generated left ideal of R . We need to show that for each R -module X the exact sequence $X \xrightarrow{v} N \rightarrow 0$ splits. Choose any injective embedding $0 \rightarrow X \xrightarrow{u} I$. Then $0 \rightarrow X' \rightarrow I \rightarrow I/X' \rightarrow 0$ is exact with $X' = \ker u$. By [3], lemma 4. 22, it is sufficient to show that any $f: N \rightarrow I/X'$ is liftable. f extends to $g: R^n \rightarrow I/X'$ by the coflatness of I/X' . g lifts to $h: R^n \rightarrow I$ since R^n is projective. It is easy to see that the restriction of h on N has X as its range. Thus there exists a $k: N \rightarrow X$ which splits v .

(c) \Rightarrow (a). Suppose I'' is a quotient module of an injective module I . To check for I'' the condition given in prop. 5 is straightforward.

If we introduce the notion of weak left global codimension (wlcd) for a ring via coflat resolution in an obvious way, then rings satisfying the equivalent conditions in prop. 7 are exactly those having $\text{wlcd} \leq 1$. The dual couple of wld - wlcd is therefore no longer symmetrical as its precedent ld - lcd .

References

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- [3] Rotman, J. J., An Introduction to Homological Algebra, New York, 1979.