第3卷 第1期
1983年 1月

数学研究与评论
JOURNAL OF MATHEMATICAL
RESEARCH AND EXPOSITION

Vol.3 No.1
Jan. 1983

Some Properties of M-Matrices*

Gu Dunhe (顾敦和)

(Huatung Institute of Engineeing)

Recently, Markham has proved the following property of M-matrices.

Theorem A[1] Suppose A is an M-matrix with the property that for some positive integer p, $A^p=L$, where L is a lower-triangular matrix. Then A is a lower-triangular matrix.

Clearly Theorem A is also true for an upper-triangular matrix. The purpose of this note is to generalize Theorem A to a block upper triangular matrix. If A is a real square matrix with nonpositive off-diagonal elements, each of the following is a necessary and sufficient condition for A to be an M-matrix.

(1) Each principal minor of A is positive.

(2) A is nonsingular, and $A^{-1}\geqslant 0$.

The proof of the main result of this note requires the following lemmas.

Lemma 1 Let $A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where A_{11} is an $r\times r$ matrix for $1\leqslant r<n$. If A^{-1} is an M-matrix, then A_{11} is invertible and A_{11}^{-1} is an M-matrix.

Lemma 2 Let A^{-1} be an M-matrix. If S is an arbitrary $r\times r$ principal submatrix of A, for $1\leqslant r\leqslant n$, then S is invertible and S^{-1} is an M-matrix.

The proofs of the above two lemmas are given in[2].

Theorem Suppose A is an M-matrix with the property that for some positive integer p, $A^p=U$, where U is a block upper triangular matrix. Then A is a block upper triangular matrix.

The above theorem is also true for a block lower triangular matrix.

Proof We proceed by induction. Suppose $p>1$ and let $A^p=U$. Since A is an M-matrix, A is invertible and $A^{-1}\geqslant 0$. But the matrix $(A^p)^{-1}=U^{-1}$ is a block upper triangular matrix with the same form as A^p. Letting $C=(A^{-1})^{p-1}$ and $B=A^{-1}$, we get $CB=U^{-1}$ and partition:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}=\begin{pmatrix} u_{11} & u_{12} \\ O & u_{22} \end{pmatrix}. \quad (1)$$

Then we have $C_{21}B_{11}+C_{22}B_{21}=0$. Because $C_{21}B_{11}\geqslant 0$ and $C_{22}B_{21}\geqslant 0$, the following

* *Received Dec.* 8, 1981.

results
$$C_{21}B_{11}=0 \tag{2}$$
and
$$C_{22}B_{21}=0 \tag{3}$$
are obtained. By Lemma 1, B_{11} is an invertible M-matrix. Right multiplication of (2) by B_{11}^{-1} yields $C_{21}=0$. Substituting $C_{21}=0$ to $C=(A^{-1})^{p-1}$, we get
$$(A^{-1})^{p-1}=\begin{pmatrix} C_{11} & C_{12} \\ O & C_{22} \end{pmatrix} \tag{4}$$
Taking determinant for (4), we obtain $\det(A^{-1})^{p-1}=\det C_{11}\det C_{22}$. Since $\det(A^{-1})^{p-1}\neq 0$, $\det C_{22}\neq 0$, that is, C_{22} is invertible. Left multiplication of (3) by C_{22}^{-1} yields $B_{21}=0$. Substituting the result to $B=A^{-1}$, we get
$$A^{-1}=\begin{pmatrix} B_{11} & B_{12} \\ O & B_{22} \end{pmatrix}. \tag{5}$$
By Lemmas 1 and 2, we know both B_{11} and B_{22} are invertible. Hence,
$$A=\begin{pmatrix} B_{11}^{-1} & -B_{11}^{-1}B_{12}B_{22}^{-1} \\ O & B_{22}^{-1} \end{pmatrix}. \tag{6}$$
So far we have proved that if A^p is a block upper triangular matrix with two principal diagonal blocks, then A is a block upper triangular matrix. The remaining part of the proof is simple and will not be discussed here.

The following consequences of the theorem are immediate.

Corollary 1 Suppose A is an M-matrix with the property that for some positive integer p; A^p is a block diagonal matrix. Then A is a block diagonal matrix;

Corollary 2 Suppose A is an M-matrix with the property that for some positive integer p, A^p is reducible. Then A is reducible.

Proof Since A is an M-matrix, it follows that PAP^T is an M-matrix, where P is a permutation matrix. Applying the above theorem to PAP^T, the conclusion can be immediately obtained.

Corollary 3 Suppose A is an irreducible M-matrix; and A^p is a matrix with positive principal diagonal elements and nonpositive off-diagonal elements, where p is a positive integer. Then A^p is an irreducible M-matrix.

Proof The conclusion can be obtained from Corollary 2, 3 of [3] and the above theorem.

References

[1] Markham, T. L., Two properties of M-matrices, Linear Algebra Appl. Vol.28 (1979), pp.131—134.

[2] 顾敦和，分块矩阵$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$为M-阵的特性，华东工程学院学报，1981年第1期，pp.31—38.

[3] Poole, G. and Boullion, T., A Survey on M-matrices, SIAM Review, Vol. 16 (1974) pp.419—427.

注：本文结果曾在1982.4.全国第一届代数学学术交流会分组会上报告过。(本文是报告内容的一部分)