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A Graceful Algorithm of a Class of Trees*

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Let T be a tree with ν vertices. It is said to be graceful if we can label the vertices with numbers $1,2,\cdots\cdots,\nu$ in such a manner that the differences of any two adjacent vertices will again form the set $\{1,2,\cdots,\nu-1\}$. Ringel conjectured, in1963, that every tree has a such labelling [1]. Ringel's conjecture remains unsettled. In this article, we shall investigate a class of graceful trees.

Definition The $T_\lambda^{(n)}$ is a tree with ν_n vertices. Its vertices are $v_j (j=0,1,\cdots,n+1)$; $v_{j_0 j_1\cdots j_\xi}(j_0=1,2,\cdots,n;\ j_1=1,2,\cdots,k_1;\ \cdots\cdots;\ j_\xi=1,2,\cdots,k_\xi,\xi\leqslant\lambda)$, where $\lambda,n,k_1,\cdots,k_\xi$ are arbitrary natural numbers. On a central line, vertices v_i and v_{i+1} are adjacent. Two vertices $v_{j_0 j_1\cdots j_\xi}$ and $v_{j_0 j_1\cdots j_\xi j_{\xi+1}}$ are adjacent if among them at least one vertex is outside the central line. In the other cases any two vertices are not adjacent. The number of all vertices $\nu_n=n(1+\sum_{p=1}^{\lambda}\Pi_{m=1}^{p}k_m)+2$.

Now, we give a numbering l for $T_\lambda^{(1)}$ as follows:

$$
(1)\quad \begin{cases} l(v_0)=1,\ l(v_1)=\nu_1,\ l(v_2)=2; \\ l(v_{j_1},\cdots,{}_{j_\xi})=\begin{cases} 2+\dfrac{\xi+1}{2}+\displaystyle\sum_{q=1}^{\xi}(j_q-1)(1+\sum_{p=q+1}^{\lambda}\prod_{m=q+1}^{p}k_m), & \text{if } \xi \text{ is odd;} \\ \nu_1-\dfrac{\xi}{2}-\displaystyle\sum_{q=1}^{\xi}(j_q-1)(1+\sum_{p=q+1}^{\lambda}\prod_{m=q+1}^{p}k_m), & \text{if } \xi \text{ is even.} \end{cases} \end{cases}
$$

If $n\geqslant 2$, we give a numbering l of $T_\lambda^{(n)}$ as follows:

$$
(2)\quad \begin{cases} l(v_0)=1,\ l(v_1)=\nu_n,\ l(v_2)=1+a,\ l(v_3)=\nu_n-a; \\ l(v_{1j_1,\cdots,j_\xi})=\begin{cases} 1+\dfrac{\xi+1}{2}+\displaystyle\sum_{q=1}^{\xi}(j_q-1)(1+\sum_{p=q+1}^{\lambda}\prod_{m=q+1}^{p}k_m), & \text{if } \xi \text{ is odd;} \\ \nu_n-\dfrac{\xi}{2}-\displaystyle\sum_{g=1}^{\xi}(j_g-1)(1+\sum_{p=g+1}^{\lambda}\prod_{m=g+1}^{p}k_m), & \text{if } \xi \text{ is even;} \end{cases} \\ l(v_{2j_i\cdots j_\xi})=\begin{cases} l(v_{1j_1'j_2'\cdots j_\xi'})+\nu_n-(a+1), & \text{if } \xi \text{ is odd;} \\ l(v_{1j_1'j_2'\cdots j_\xi'})-[\nu_n-(a+1)] & \text{if } \xi \text{ is even.} \end{cases} \end{cases}
$$

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where $j_i' = k_i + 1 - j_i$) is called the complementary index of $j_i, i = 1, 2, \cdots, \xi$), and $a = 1 + \Sigma_{p=1}^{\lambda} \Pi_{m=1}^{p} k_m$. when $q = \lambda$, we set $\Sigma_{p=q+1}^{\lambda} \Pi_{m=q+1}^{p} k_m = 0$. In general terms, we have

$$(3)\quad \begin{cases} l(v_{(2r)}) = 1 + ra, \ l(v_{(2r+1)}) = \nu_n - ra; \\ l(v_{(2r+1)j_1\cdots j_\xi}) = \begin{cases} l(v_{1j_1\cdots j_\xi}) + ra, & \text{if } \xi \text{ is odd}; \\ l(v_{1j_1\cdots j_\xi}) - ra, & \text{if } \xi \text{ is even}; \end{cases} \\ l(v_{(2r+2)j_1\cdots j_\xi}) = \begin{cases} l(v_{2j_1\cdots j_\xi}) - ra, & \text{if } \xi \text{ is odd}; \\ l(v_{2j_1\cdots j_\xi}) + ra, & \text{if } \xi \text{ is even}, \end{cases} \end{cases}$$

$$(r = 0, 1, \cdots, \left[\frac{n}{2}\right] - 1).$$

when n is odd, then we further have

$$(4)\quad \begin{cases} l(v_{n-1}) = 1 + \frac{1}{2}(n-1)a, \ l(v_n) = \nu_1 + \frac{1}{2}(n-1)a, l(v_2) = 2 + \frac{1}{2}(n-1)a; \\ l(v_{nj_1\cdots j_\xi}) = \begin{cases} \frac{1}{2}(n-1)a + 2 + \frac{\xi+1}{2} + \sum_{q=1}^{\xi}(j_q - 1)(1 + \sum_{p=q+1}^{\lambda} \prod_{m=q+1}^{p} k_m), \text{if } \xi \text{ is odd}; \\ \frac{1}{2}(n-1)a + \nu_1 - \frac{\xi}{2} - \sum_{q=1}^{\xi}(j_q - 1)(1 + \sum_{p=q+1}^{\lambda} \prod_{m=q+1}^{p} k_m), \text{ if } \xi \text{ is even}. \end{cases} \end{cases}$$

We can prove that the given numbering l is a graceful algorithm of $T_\lambda^{(n)}$. Honce $T_\lambda^{(n)}$ is graceful. According to the definition in the literature [2], further, we can prove that the numbering l for $T_\lambda^{(n)}$ is interlaced when n is even.

Let T_λ^* denote the tree left over when we have taken away v_0 and v_2 from $T_\lambda^{(1)}$. The number of its vertices then is $a = 1 + \sum_{p=1}^{\lambda} \prod_{m=1}^{p} k_m$. Similarly, we can give a graceful numbering l for T_λ^* as follows:

$$(5)\quad \begin{cases} l(v_1) = a; \\ l(v_{1j_1\cdots j_\xi}) = \begin{cases} \frac{\xi+1}{2} + \sum_{q=1}^{\xi}(j_q - 1)(1 + \sum_{p=q+1}^{\lambda} \prod_{m=q+1}^{p} k_m), \text{ if } \xi \text{ is odd}; \\ a - \frac{\xi}{2} - \sum_{q=1}^{\xi}(j_q - 1)(1 + \sum_{p=q+1}^{\lambda} \prod_{m=q+1}^{p} k_m), \text{ if } \xi \text{ is even}. \end{cases} \end{cases}$$

When $k_1 = k_2 = \cdots = k_\lambda = k$ in (5), we have a graceful numbering for the complete k-any trees [5]. Further, when $k = 2$, we have an answer to Cahit's question [1].

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