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Inductive Proof of a Saddle Point Theorem*

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Abstract. Many proofs have been published for the minimax theorem, and all the published inductive proofs have been indirect ones. It has been pointed out that a direct inductive proof is needed, especially for instructional purposes, since indirect proofs are more or less implicit in nature. Such a direct proof is given in [4]. Now the minimax theorem can be stated equivalently in terms of saddle point. And it is the object of the present paper to give a direct inductive proof for the saddle point version of this theorem.

In this paper, we present a direct inductive proof for the saddle point theorem of zero sum two person games with mixed strategies.

Theorem. Let $A=(a_{ij})$ be an arbitrary $m\times n$ matrix. Let S_m and S_n be respectively set of points $x=(x_1,\cdots,x_m)$ and $y=(y_1,\cdots,y_n)$ with

$$x_i\geqslant 0,\ \ i=1,\cdots,m,\quad \sum_{i=1}^{m}x_i=1,\qquad y_j\geqslant 0,\ \ j=1,\cdots,n,\quad \sum_{j=1}^{n}y_j=1.$$

Then there exist $x^*=(x_1^*,\cdots,x_m^*)\in S_m$ and $y^*=(y_1^*,\cdots,y_n^*)\in S_n$ such that

$$xAy^{*t}\leqslant x^*Ay^{*t}\leqslant x^*Ay^t$$

for all $x\in S_m$ and all $y\in S_n$.

Proof. It is obvious that the theorem is true for $m=n=1$. Assume that the theorem holds for all $(m'<m,\ n)$, let us prove that it is true for (m,n). (In a similar manner it can be shown that if the theorem holds for all $(m,\ n'<n)$, then it is true for (m,n).)

Suppose that

$$\min_{y\in S_n}\max_{1\leq i\leq m}\sum_{j=1}^{n}a_{ij}y_j=\max_{1\leq i\leq m}\sum_{j=1}^{n}a_{ij}y_j^*=v. \tag{1}$$

Then

$$A_{i.}y^{*t}=\sum_{j=1}^{n}a_{ij}y_j^*\leqslant v,\ \ i=1,\cdots,m. \tag{2}$$

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If equality holds in (2) for all $i=1,\cdots,m$, as well as in an analogous greater-than-or-equal formula corresponding to the max min operation for all $j=1,\cdots,n$, the validity of the theorem is easily proved. Thus we may assume without loss of generality that

$$A_{i.}y^{*t}=\sum_{j=1}^{n}a_{ij}y_j^*=v,\quad i=1,\cdots,m',\tag{3}$$

$$A_{i.}y^{*t}=\sum_{j=1}^{n}a_{ij}y^*_j<v,\quad i=m'+1,\cdots,m,\tag{4}$$

where $m'<m$.

Now consider the reduced matrix game (m',n). By the inductive hypothesis, there exist $x'=(x_1^*,\cdots,x_{m'}^*)\in S_{m'}$ and $y'\in S_n$ such that

$$xAy'^t\leqslant x'Ay'^t\leqslant x'Ay^t\tag{5}$$

for all $x\in S_{m'}$ and all $y\in S_n$. (Here A is the reduced $m'\times n$ matrix.) In other words, (x',y') is a saddle point of the game (m',n). It follows that

$$\max_{x\in S_{m'}}xAy'^t=x'Ay'^t=\min_{y\in S_n}x'Ay^t,$$

or

$$\max_{1\leqslant i\leqslant m'}A_{i.}y'^t=x'Ay'^t=\min_{1\leqslant j\leqslant n}x'A_{.j}.\tag{6}$$

Our next task is to show that $x'Ay'^t\geqslant v$. For this purpose, let

$$y''=ay'+(1-a)y^*\in S_n,\quad 0<a<1.\tag{7}$$

Then

$$A_{i.}y''^t=aA_{i.}y'^t+(1-a)A_{i.}y^{*t},\quad i=1,\cdots,m'.\tag{8}$$

Taking $\max\limits_{1\leqslant i\leqslant m'}$ on both sides of (8) and utilizing (6) and (3), we obtain

$$\max_{1\leqslant i\leqslant m'}A_{i.}y''^t\leqslant a\max_{1\leqslant i\leqslant m'}A_{i.}y'^t+(1-a)\max_{1\leqslant i\leqslant m'}A_{i.}y^{*t}=ax'Ay'^t+(1-a)v.\tag{9}$$

Now for the y'' in (7) we have by (4) and the continuity of the functions involved that

$$A_{i.}y''^t<v,\qquad i=m'+1,\ldots\ldots,m\tag{10}$$

if a is sufficiently small. But

$$\max_{1\leqslant i\leqslant m}A_{i.}y''^t\geqslant\min_{y\in S_n}\max_{1\leqslant i\leqslant m}A_{i.}y^t=v\tag{11}$$

by (1); It follows from (10) and (11) that

$$\max_{1\leqslant i\leqslant m'}A_{i.}y''^t\geqslant v.\tag{12}$$

Hence (9) implies $v\leqslant ax'Ay'^t+(1-a)v$, or $ax'Ay'^t\geqslant v-(1-a)v=av$, or

$$x'Ay'^t\geqslant v,\tag{13}$$

as is to be proved.

Finally, consider the point $x^* = (x_1^*, \cdots, x_{m'}^*, 0, \cdots, 0)$ of S_m. We are going to show that the strategy pair (x^*, y^*) is a saddle point of the game (m, n).

We have, by (3) and (4),

$$x^* A y^{*t} = \sum_{i=1}^{m} x_i^* A_{i.} y^{*t} = v, \tag{14}$$

where $x_i^* = 0$, $i = m'+1, \cdots, m$.

For all $x \in S_m$ we have $xAy^{*t} = \sum_{i=1}^{m} x_i A_{i.} y^{*t} \leqslant v$ by (2). Hence

$$xAy^{*t} \leqslant x^* A y^{*t} \tag{15}$$

for all $x \in S_m$. And for all $y \in S_n$ we have

$$x^* A y^t = \sum_{j=1}^{n} x^* A_{.j} y_j = \sum_{j=1}^{n} x' A_{.j} y_j = x' A y^t \geqslant x' A y'^t \geqslant v$$

by (5) and (13). (It is to be noted that in the extreme left hand side the dimensionality of the matrix A is $m \times n$, while the last two $A's$ are $m' \times n$ matrices.) Hence

$$x^* A y^{*t} \leqslant x^* A y^t \tag{16}$$

for all $y \in S_n$.

(15) and (16) show that (x^*, y^*) is a saddle point of the matrix game (m, n) and the proof of the theorem is completed.

References

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