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A Simple Proof for a Theorem of Kelisky and Rivlin*

Chang Geng-zhe (常庚哲) *and Shan Zun* (单墫)

(China University of Science and Technology)

Theorem (Kelisky and Rivlin) Let $f(x)$ be a function defined in $[0,1]$ and $B_n(f(x))=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}$ be the nth Bernstein polynomial of $f(x)$. Then $\lim\limits_{l\to+\infty} B^l(f(x))=f(0)+(f(1)-f(0))x$.

Proof We can assume $f(0)=0$. Let $\phi_i(x)$ and $\psi_i(x)$ $(i=1,2,\cdots,n)$ be Bernstein basis polynomials and Bezier basis polynomials respectively; Let $n\times n$ matrices

$$T=\begin{pmatrix}1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ & & & & 1\end{pmatrix} \quad \text{and} \quad B=\left(\phi_i\left(\frac{j}{n}\right)\right)=\begin{pmatrix} C & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ * & 1\end{pmatrix}=V^{-1}\begin{pmatrix}1 & \\ & J_{n-1}\end{pmatrix}V,$$

where V is a matrix which reduces B to Jordan form.

Each column sum of the matrix C is less than 1. Hence the absolute values of the characteristic roots of B are less than 1 except the simple root $\lambda=1$; Thus

$$\lim_{l\to+\infty} T^{-1}\ B^l\ T=T^{-1}\ V^{-1}\ \lim_{l\to+\infty}\begin{pmatrix}1 & \\ & J_{n-1}^l\end{pmatrix}VT=T^{-1}\ V^{-1}\begin{pmatrix}1 & \\ & O_{n-1}\end{pmatrix}VT=(a_ib_j)$$

where $(a_1,a_2,\cdots,a_n)'$ is the first column of $T^{-1}\ V^{-1}$ and $(b_1,b_2,\cdots,b_n)$ is the first row of VT;

From the properties of Bezier basis polynomials we know that $T^{-1}BT=\left(\psi_i\left(\frac{j}{n}\right)-\psi_i\left(\frac{j-1}{n}\right)\right)$ is a d.s. (doubly stochastic) matrix. Hence the matrix $(a_i\,b_j)$ is also a d.s. matrix. It follows that $a_i\ b_j=\frac{1}{n}$. Thus

$$\lim_{l\to+\infty} B^l(f(x))=\lim_{l\to+\infty}\left(f\left(\frac{1}{n}\right),f\left(\frac{2}{n}\right),\cdots,\ f\left(\frac{n}{n}\right)\right)B^{l-1}(\phi_1(x),\ \phi_2(x),\cdots,\phi_n(x))'$$

$$=\left(f\left(\frac{1}{n}\right),f\left(\frac{2}{n}\right),\cdots,\ f\left(\frac{n}{n}\right)\right)T\ \lim_{l\to+\infty}(T^{-1}\ B^{-1}\ T)T^{-1}(\phi_1(x),\ \phi_2(x),\cdots,\phi_n(x))'$$

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$$=\left(f\left(\frac{1}{n}\right),\ f\left(\frac{2}{n}\right),\cdots,\ f\left(\frac{n}{n}\right)\right)\begin{pmatrix}1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & & \\ & & & 1 & -1 \\ & & & & 1\end{pmatrix}\begin{pmatrix}\frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \cdots & \cdots & & \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n}\end{pmatrix}\begin{pmatrix}1 & 1 & \cdots & 1 \\ & 1 & \cdots & 1 \\ & & \ddots & \\ & & & 1\end{pmatrix}\begin{pmatrix}\phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_n(x)\end{pmatrix}$$

$$=\frac{1}{n}\left(f\left(\frac{1}{n}\right),\cdots,\ f\left(\frac{n}{n}\right)\right)\begin{pmatrix} & & \\ & O & \\ & & \\ 1 & 1 \cdots & 1\end{pmatrix}\begin{pmatrix}\psi_1(x) \\ \vdots \\ \psi_n(x)\end{pmatrix}$$

$$=\frac{1}{n}\left(f\left(\frac{1}{n}\right),\quad f\left(\frac{2}{n}\right),\cdots,\ f\left(\frac{n}{n}\right)\right)(0,\cdots,\ 0,\ nx)'=f(1)x.$$

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