第3卷 第1期
1983年 1月

数 学 研 究 与 评 论
JOURNAL OF MATHEMATICAL
RESEARCH AND EXPOSITION

Vol.3 No.1
Jan. 1983

The Asymptotically Optimal Empirical Bayes Estimation about a Class of Uniform Distribution*

Wei Leisheng (韦来生) *Fang Zhaoben* (方兆本) *Li Jinping* (李金平)

(China University of Science and Technology)

In this note, we propose a squared error loss empirical Bayes estimator of θ based on past experiences and a present observation X which has conditional distribution $U(\theta, c\theta+b)$, where b is an arbitary constant when $c>1$; $b>c$ when $c=1$, $\theta\in\Omega=(-\frac{b}{c-1},\infty)$. When unkown prior $G(\theta)$ of θ belongs to the family $\mathscr{F}=\{G: \int_\Omega \theta^2 dG(\theta)<\infty\}$, our estimator is asymptotically optimal (see [1]).

Let $K(x)$ and $k(x)$ be marginal distribution and density of r. v. X. It is easily seen that

$$k(x)=\int_{\frac{x-b}{c}}^{x}\frac{1}{(c-1)\theta+b}dG(\theta), K(x)=G\left(\frac{x-b}{c}\right)+xk(x)-\int_{\frac{x-b}{c}}^{x}\frac{\theta}{(c-1)\theta+b}dG(\theta). \quad (1)$$

Let $\phi_G(x)$ denote the Bayes estimator of θ versus G, we have $\phi_{\bullet}(x)=\frac{1}{c}[\psi(x)+(x-b)]$, where $\psi(x)=[G(x)-K(x)]/k(x)$ (undefined ratios are taken to be zero). Let R_G be the Bayes risk ver sus G, i. e. $R_G=E_{(x;\theta)}(\phi_G(x)-\theta)^2$. Let $(X_1, X_2, \cdots, X)$ be a sequence. of i. i. d. r. v, according to $K(x)$. Let $\tilde{P}^\infty$ and P be the product measure on the space of sequence $(X_1, X_2\cdots)$ and $(X_1, X_2, \cdots(X,\theta))$ respectively.

Based on $x_1,\cdots,x_n$ (historical date) and x (present sample) we estimate $K(x)$ and $k(x)$ by

$$K_n(x)=\frac{1}{n}\sum_{i=1}^{n}I_{[x_i\le x]}, \quad k_n(x)=\frac{K_n(x+h)-K_n(x)}{h}, \quad (2)$$

where $I_{[A]}$ represents the indication function of the event A. When $c>1$, (1) implies

$$G\left(\frac{x-b}{c}\right)-\frac{1}{c}G(x)=\frac{c-1}{c}K(x)-\frac{1}{c}[(c-1)x+b]k(x). \quad (3)$$

Substituting $cx+b$ for x one by one, we come to

$$G(x)=(c-1)\sum_{j=1}^{\infty}\frac{1}{c^j}K(c^jx+q_{j-1}b)-[(c-1)x+b]\sum_{j=1}^{\infty}k(c^jx+q_{j-1}b), \quad q_j=\sum_{l=0}^{j}c^l. \quad (4)$$

Received Sept. 5, 1981.

Similarly, when $c=1$

$$G(x)=1-b\sum_{j=1}^{\infty}k(x+jb). \tag{5}$$

From this, estimate $G(x)$ by

$$G_n(x)=\delta(c,1)+(c-1)\sum_{j=1}^{\infty}\frac{1}{c^j}K_n(c^jx+q_{j-1}b)-[(c-1)x+b]\sum_{j=1}^{\infty}k_n(c^jx+q_{j-1}b), \tag{6}$$

where $\delta(c,1)=\begin{cases}1, & c=1,\\ 0, & c>1.\end{cases}$ Since $0\leqslant\psi(x)\leqslant(c-1)x+b$, estimate $\psi(x)$ by

$$\psi_n(x)=\min\left\{(c-1)x+b,\ \left(\frac{G_n(x)-K_n(x)}{k_n(x)}\right)^+\right\}, \tag{7}$$

where $(g)^+=\max(g,0)$. Define

$$\phi_n(x)=\frac{1}{c}(\psi_n(x)+x-b). \tag{8}$$

Let $R_n=E_{(x_1,\cdots;(x,\theta))}(\phi_n(x)-\theta)^2$ (Bayes risk of ϕ_n). If $\lim\limits_{n\to\infty}R_n=R_G$, $\forall G\in\mathscr{F}$, $\phi_n(x)$ is an a. o. EB (asymptotically optimal Empirical Bayes) estimator versus $\mathscr{F}$. Our main result is

Theorem If $nh^2\to\infty, h\to 0$, EB estimator $\phi_n(x)$ of θ (defined by(8)) is a. o. versus $\mathscr{F}$.

Corollary For Uniform distribution family $U(\theta,\theta+b)$, $\phi_n(x)$ is a. o, versus any prior G.

Lemma 1 If $nh^2\to\infty, h\to 0$, for each x, $G_n(x)\to G(x)(\tilde{p}^\infty)$ (i. e. convergence in probability measure $\tilde{p}^\infty$)

Proof From (6),

$$G_n(x)=\delta(c,1)+(c-1)\frac{1}{n}\sum_{i=1}^{n}Y_i-\frac{1}{n}[(c-1)x+b]h^{-1}\sum_{i=1}^{n}Z_i,$$

where $Y_i=\sum_{j=1}^{\infty}\frac{1}{c^j}I_{[x_i\leqslant c^jx+q_{j-1}b]}$, $Z_i=\sum_{j=1}^{\infty}I_{[c^jx+q_{j-1}b<x_i\leqslant c^jx+q_{j-1}b+h]}$, $Y_1,\cdots,Y_n$, i. i. d, $Z_1,\cdots,Z_n$, i. i. d, When $c>1$, we have $Y_i\leqslant\frac{1}{c-1}$. We choose h such that $0<h<(c-1)x+b$, then $Z_i\leqslant 1$, thus

$$E\left(\frac{1}{n}\sum_{i=1}^{n}Y_i\right)=\sum_{j=1}^{\infty}\frac{1}{c^j}K(c^jx+q_{j-1}b), E\left(\frac{1}{n}\sum_{i=1}^{n}Z_i\right)=\sum_{j=1}^{\infty}[K(c^jx+q_{j-1}b+h)-K(c^jx+q_{j-1}b)].$$

Then, it is easily shown that

$$\lim_{h\to 0}h^{-1}\sum_{j=1}^{\infty}[K(c^jx+q_{j-1}b+h)-K(c^jx+q_{j-1}b)]=\sum_{j=1}^{\infty}k(c^jx+q_{j-1}b). \tag{9}$$

Denote $E(x_1,x_2,\cdots)$ by $\tilde{E}$, we have

$$\tilde{E}(G_n(x)-G(x))^2\leqslant 2\tilde{E}(G_n(x)-\tilde{E}G_n(x))^2+2\tilde{E}(\tilde{E}G_n(x)-G(x))^2\triangleq 2(I_1+I_2).$$

From(9), $I_2 \to 0$, as $h \to 0$; $I_1 \leqslant \frac{2(c-1)^2}{n} VarY_1 + \frac{2[c-1)x+b]^2}{nh^2} VarZ_1 \to 0$, as $nh^2 \to \infty$. For $c=1$, it is trival.

Lemma 2 If $nh^2 \to \infty, h \to 0$, *then* $\psi_n - \psi \to 0 (P)$

Proof Let $x \in A = \{x | k(x) > 0\}$. By Tchebichev inequality $k_n(x) - h^{-1}(K(x+h) - K(x)) \to 0$ $(\tilde{p}^\infty)$. Since $h^{-1}(K(x+h) - K(x)) \to k(x)$ *a. e.* $\tilde{p}^\infty$. $k_n(x) \to k(x)$ $(\tilde{p}^\infty)$. By Glivenko-Cantelli theorem (See [2]), $K_n(x) - K(x) \to 0$ $(\tilde{p}^\infty)$, Denote $S_n = [G_n(x) - K_n(x)]/k_n(x)$. Since $k(x) > 0$ in A, $S_n \to \psi(x)$ $(\tilde{p}^\infty)$. We can also prove that $S_n^+ - \psi(x) \to 0 (\tilde{p}^\infty)$. Since $\tilde{p}^\infty(|S_n^+ - \psi| > \varepsilon) \geqslant \tilde{p}^\infty(|\psi_n - \psi| \geqslant \varepsilon)$, $\psi_n - \psi \to 0$ $(\tilde{p}^\infty)$. Thus, by $P(A) = 1$, Fubini theorem, and the dominated convergence theorem. $\psi_n - \psi \to 0$ (P).

Finally, we prove the theorem. Since $G \in \mathscr{F}$, $E_{(x,\theta)}(x^2) < \infty$, $R_G < \infty$, $R_n - R_G = E_{(x_1,\cdots,x_n,x)}(\phi_n - \phi_G)^2 = \frac{1}{c^2} E_{(x_1,\cdots x_n,x)}(\psi_n - \psi)^2$ (See[3]). Since $|\psi_n - \psi|^2 \leqslant 8\{(c-1)^2 x^2 + b^2\}$, by Lemma 1,2 and the dominated convergence theorem, we have $R_n - R_G \to 0 (h \to 0, nh^2 \to \infty)$. We omit the proof of corollary; When $b=1$, Theorem 3,1 of [1] is a particular case of our corollary.

Acknowledgment. We wish to thank Professor Chen Xiru and Comrade Zhao Linchen for their guidance.

Reference

[1] Fox, R. J., (1978), *Ann.Statist.*, Vol. 6, No. 4, 846-853.
[2] Michel Loéve (1963), Probability Theory (3rd,ed).
[3] Singh, R, S. (1979), *Ann, Statist.*, Vol. 7, No. 4, 890-902.