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# Self-duality of Multiple Objective Mathematical Programming

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## 1. Introduction

In this paper, we discuss the self-duality of multiple objective mathematical programming. The main purpose is to extend the self-duality of single objective mathematical programming given by papers [1] and [2] to the case of multiple objective, to establish the self-duality of multiple objective linear and quadratic programming for efficient solution, weak efficient solution and properly efficient solution.

## 2. Self-duality of multiple objective linear programming.

The single objective linear programming in paper [1] may be extended as following multiple objective linear programming:

$$(\mathrm{VLP})\begin{cases}\min\ Bx\\ Ax\geqq -B^T\lambda\\ x\geqq 0\end{cases}$$

where $A$ and $B$ are $n$-order square matrices, $A$ is skew-symmetric, $x$ and $\lambda$ are $n$-dimensional column vectors, $T$ denote transport, $\lambda\in\Lambda^+$ or $\Lambda^{++}$, there $\Lambda^+=\{\lambda\,|\,\lambda_i\geqq 0, \sum_{i=1}^n\lambda_i=1\}$, $\Lambda^{++}=\{\lambda\,|\,\lambda_i>0, \sum_{i=1}^n\lambda_i=1\}$.

Let (VLP)$(\bar\lambda)$ denote such (VLP) when given $\lambda=\bar\lambda$ in (VLP).

We can prove following results:

**Theorem 1** (VLP) is self-dual for its efficient solutions or weak efficient solutions or properly efficient solutions.

**Theorem 2** Let $\bar\lambda\in\Lambda^+$(or $\Lambda^{++}$), if (VLP)$(\bar\lambda)$ have finite feasible solutions, then (VLP)$(\bar\lambda)$ certainly have weak efficient solutions (or properly efficient solutions). Further, for an arbitrary weak efficient solution (or properly efficient solution) $\bar x$ of (VLP) $(\bar\lambda)$, we have $\bar\lambda^T(B\bar x)=0$.

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**Corollary 1** Let $\bar{\lambda}\in\Lambda^+$, and define $I(\bar{\lambda})=\{i|\bar{\lambda}_i>0,\ 1\leqq i\leqq n\}$. If for $\forall i\in I(\bar{\lambda})$ the row vectors $b_i\geqq 0$ in $B$, then for arbitrary weak efficient solution $\bar{x}$ of (VLP)$(\bar{\lambda})$, we have $b_i\bar{x}=0$ for $i\in I(\bar{\lambda})$.

**Corollary 2** Under the assumptions of corollary 1, if there exists at least one $i_0\in I(\bar{\lambda})$ such that $b_{i_0}>0$, then all weak efficient solutions of (VLP)$(\bar{\lambda})$ are zero solution.

If all $b_ix(i=1,2,\cdots,n)$ are monotone strictly, then theorem 2, corollary 1 and 2 are also true for efficient solutions of (VLP)$(\lambda)$.

**Corollary 3** Let $\bar{\lambda}\in\Lambda^{++}$. If all $b_i\geqq 0(i=1,2,\cdots,n)$, then for arbitrary properly efficient solution $\bar{x}$ of (VLP)$(\bar{\lambda})$, we have $B\bar{x}=0$.

**Corollary 4** Under the assumptions of corollary 3, if there exists at least one $i_0(1\leqq i_0\leqq n)$ such that $b_{i_0}>0$, then all properly efficient solutions of (VLP)$(\bar{\lambda})$ are zero solution.

For the dual programming of (VLP), we can also establish those preperties which similar to theorem 2 and corollary 1-4 for (VLP) in above.

### 3. Self-duality of multiple objective quadratic programming

The single objective quadratic programming in paper [2] may be extended as following multiple objective quadratic programming:

$$(\mathrm{VQP})\begin{cases}\min\ F(x)=(f_1(x),f_2(x),\cdots,f_n(x))^T\\ \sum\limits_{i=1}^{n}\lambda_iA_ix\geqq-\sum\limits_{i=1}^{n}\lambda_ip_i\\ x\geqq 0,\end{cases}$$

where each $f_i(x)=x^TAx+p_i^Tx$, $x$ and each $p_i$ are $n$-dimensional column vectors, $(\lambda_1,\lambda_2,\cdots,\lambda_n)\in\Lambda^+$ or $\Lambda^{++}$, and each $A_i$ are $n$-order square matrix. We assume each $A_i$ are positive semi-difinite, but the convex combination $\sum\limits_{i=1}^{n}\lambda_iA_i$ is positive difinite.

We can prove following results:

**Theorem 3** (VQP) is self-dual for its weak efficient solutions or properly efficient solutions.

**Theorem 4** For $\forall\bar{\lambda}\in\Lambda^+$(or$\Lambda^{++}$), the (VQP)$(\bar{\lambda})$ certainly have weak efficient solutions (or properly efficient solutions). Further, for an arbitrary weak efficient solution (or properly efficient solution) $\bar{x}$ of (VQP)$(\bar{\lambda})$, we have $\bar{\lambda}^TF(\bar{x})=0$.

**Corollary 5** Let $\bar{\lambda}\in\Lambda^+$, and define $I(\bar{\lambda})=\{i|\bar{\lambda}_i>0,\ 1\leqq i\leqq n\}$. If $p_i\geqq 0$ for $\forall i\in I(\bar{\lambda})$, then for arbitrary weak efficient solution $\bar{x}$ of (VQP)$(\bar{\lambda})$, we have $f_i(\bar{x})=0$ for $i\in I(\bar{\lambda})$.

**Corollary 6** Under the assumptions of corollary 5, if there exists at least one $i_0 \in I(\bar{\lambda})_0$ such that $A_{i_0}$ is positive definite, then all weak efficient solutions of (VQP)$(\bar{\lambda})$ are zero solution.

If we take assumptions in addition that all $A_i$ are positive definite, then the theorem 3 and 4, corollary 5 and 6 are also true for efficient solutions of (VQP)$(\bar{\lambda})$.

**Corollary 7** Let $\bar{\lambda} \in \Lambda^{++}$. If all $p_i \geqq 0$ $(i=1,2,\cdots,n)$, then for arbitrary properly efficient solution $\bar{x}$ of (VQP)$(\bar{\lambda})$, we have $F(\bar{x})=0$.

**Corollary 8** Under the assumptions of corollary 7, if there exists at least one $i_0$ $(1 \leqq i_0 \leqq n)$ such that $A_{i_0}$ is positive definite, then all properly efficient solutions of (VQP)$(\bar{\lambda})$ are zero solution.

For the dual programming of (VQP). we can also establish properties similar to theorem 4 and corollary 5-8.

## References

[1] Duffin, R. J., Infinite programs, Linear inequalties and related systems, Princeton University Press, 1956.

[2] Dorn,, W. S. Self-dual quadratic programs, *J. Soc. Indust. Appl. Math.*, 9, 1, 51-54, 1961.