第3卷 第1期
1983年 1月

数 学 研 究 与 评 论
JOURNAL OF MATHEMATICAL
RESEARCH AND EXPOSITION

Vol.3 No.1
Jan. 1983

A Survey of the Whittaker-Shannon Sampling Theorem and Some of its Extensions*

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Abstract

The aim of this paper is to present a survey of results concerning the Whittaker-Kotel′nikov-Raabe-Shannon-Someya sampling theorem and its various extensions obtained at Aachen since 1977. This theorem, basic in communication engineering, is often called the cardinal interpolation series theorem in mathematical circles. The interconnections of the sampling theorem (in the setting of Paley-Wiener space) with the theory of Fourier series and integrals are examined. Emphasis is placed upon error analysis, including the aliasing, round-off (or quantization), and time jitter errors. Some new error estimates are given, others are improved: many of the proofs are reduced to a common structure. Both deterministic and probabilistic methods are employed. whereas these results are worked out in detail, the paper also contains a brief discussion of some of the various generalizations.

1. Introduction and History

The well-known Shannon (1940/49)[58] sampling theorem which plays a basic role in communication, control theory and data processing, states that every real-valued signal function $f(t)$ that is bandlimited to $[-\pi W, \pi W]$, $W>0$, can be completely reconstructed from its values (samples $f(k(W))$ taken at the nodes k/W equally spaced apart on the real axis $\mathbf{R}$, in terms of

$$f(t)=\sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right)\frac{\sin\pi(Wt-k)}{\pi(Wt-k)}=\sin(\pi Wt)\sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right)\frac{(-1)^k}{\pi(Wt-k)}\qquad(t\in\mathbf{R}) \tag{1.1}$$

*Received May 19, 1982.

$\left(\sum_{k=-\infty}^{\infty}\right.$ being understood as $\lim_{n,m\to\infty}\sum_{k=-n'}^{m}$, usually with $\left.m=n\right)$. The latter form of this series recalls to mind that it had been considered much earlier by the mathematicians C. de La Vallée Poussin (1908)[76], E. T. Whittaker (1915)[81], W. L. Ferrar (1925)[32] and J. M. Whittaker (1929/35)[80,81]; the sampling series was called Whittaker's cardinal (interpolation) series by them, or cardinal function provided this series converges.

Returning to the engineering literature, V. A. Kotel'nikov (1933)[41] and H. Raabe (1939)[51] had considered the theorem earlier than, and I. Someya (1949)[60] parallel to C. Shannon. In any case, between 1950 and 1975 at least 250 articles dealing with various aspects of the sampling theorem, written by about 170 different authors, appeared in engineering journals (see the survey paper by A. J. Jerri [40] and the historical report by H. D. Lüke [45]).

But from the point of view of pure mathematics there is comparatively only a small number of papers on the subject, e. g. by G. H. Hardy (1941)[35], L. L. Campbell (1964)[26], J. L. Brown Jr. (1967)[7], I. J. Schoenberg (since 1969), cf. [55,56], R. Kress (1970)[43,44], J. McNamee-F. Stenger-E. L. Wnitney (1971)[48], R. P. Boas (1972)[5], H. Pollard-O. Shisha (1972)[50]. Boas-Pollard (1974)[6], L. B. Sofman (1974)[59], V. I. Buslaev-A. G. Vituškin (1974)[11], Vituškin (1976)[77], Stenger (1976)[72],[46], J. R. Higgins (1976)[36], R. Gervais-Q. I. Rahman-G. Schmeisser (1978)[33,34], who followed up the work of the four pioneering mathematicians mentioned. Although the material should be a standard topic in books on Fourier analysis, just the books by Boas (1954)[4], H. Schönhage (1971)[57], H. Dym-H. P. McKean (1972)[28], J. R. Higgins (1977)[37], H. Triebel (1977)[75] and R. M. Young (1980)[83] seem to cover the sampling theorem.

The aim of this paper*) is to try to draw the sampling theorem to the attention of a wider group of mathematicians, for it indeed belongs to the interdisciplinary domain of Fourier analysis, interpolation, approximation and communication engineering. For this purpose I will mainly report on a few of the developments in connection with this theorem at Aachen since 1977. These were carried out chiefly by Dr. W. Splettstösser, but also by Drs. R. L. Stens, G. Wilmes, W. Engels and the present author. At the same time most of the results are presented in a sharpened form, from a fresh point of view, or with new proofs. As a matter of fact, all but one of the proofs have been reduced to a common structure. Theorems 7 and 9b) seem to be new.

*) This paper is a modified and an enlarged version of Butzer [13] with new material and references.

Let me first look at the hypotheses of the theorem. It is generally assumed that f belongs to $C(\mathbf{R})$ (= class of functions which are uniformly continuous and bounded on the real axis $\mathbf{R}$) and to $L(\mathbf{R})$ (= class of functions which are absolutely integrable over $\mathbf{R}$ in Lebesgue's sense). That $f(t)$ is bandlimited to $[-\pi W, \pi W]$, some $W>0$, means that the Fourier transform of $f\in L(\mathbf{R})$, namely $f^{\wedge}(v) := (1/\sqrt{2\pi})\cdot\int_R f(u)e^{ivu}du$, $v\in\mathbf{R}$, vanishes for all $|v|>\pi W$. In that case the Fourier inversion theorem gives

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\pi W}^{\pi W} f^{\wedge}(v)e^{itv}dv. \qquad (t\in\mathbf{R}). \tag{1.2}$$

The sampling theorem may now be stated more precisely.

Theorem 1 *If $f\in C(\mathbf{R})\cap L(\mathbf{R})$, and f is bandlitmied to $[-\pi W, \pi W]$, then the representation* (1.1) *holds for each $t\in\mathbf{R}$, the series being absolutely and uniformly convergent on $\mathbf{R}$.*

If $f\in C(\mathbf{R})\cap L(\mathbf{R})$, then $f\in L^2(\mathbf{R})$ (= class of functions quadratically integrable over $\mathbf{R}$). In this regard, the Paley-Wiener theorem states that any $f\in L^2(\mathbf{R})$ has the representation (1.2) (or, more generally, $f(t) = (1/\sqrt{2\pi})\int_{-\pi W}^{\pi W} g(v)e^{itv}dv$ with $g\in L^2[-\pi W, \pi W]$) iff f is the restriction to $\mathbf{R}$ of an entire function F of exponential type πW, i. e.,

$$|F(z)|\leqslant e^{\pi W|y|}\|F\|_C \qquad (z=x+iy\in C),$$

where $\|F(\cdot)\|_C := \|F\|_C := \sup_{t\in R}|F(t)|$. The class of such functions f in $L^2(\mathbf{R})$ will be denoted by $B_{\pi W}$; see R. M. Young [83, Chap. 2]. So the hypothesis of Theorem 1 may be replaced by $f\in B_{\pi W}$.

Concerning the series (1.1) itself, namely the cardinal function, it is of interest that it interpolates f at the nodes $t=k/W$, $k\in\mathbf{Z}$, just because

$$\mathrm{si}\{\pi(Wt-k)\} := \frac{\sin\pi(Wt-k)}{\pi(Wt-k)} = \begin{cases} 1, & t=k/W \\ 0, & t=1/W, k\neq 1\in z. \end{cases}$$

This leads to a "formal" proof of (1.1): if it would be known that the sum in the first series of (1.1), regarded as a convolution sum of f and the si-function, be commutative; then it would be equal to

$$\sum_{k=-\infty}^{\infty} f\left(t-\frac{k}{W}\right)\frac{\sin\pi k}{\pi k} = f(t). \qquad (t\in\mathbf{R})$$

2. Sampling Expansions of Non-Bandlimited Functions

Practice demands that one also tries to consider representations of type (1.1) for duration-limited functions, i. e., for functions $f\in C(\mathbf{R})$ such that $f(t)=0$ for all $|t|>T$, some $T>0$. Since such functions cannot be simultaneously bandlimited (unless they vanish everywhere), in view of the Paley-Wiener theorem and the identity theorem for holomorphic functions, one needs to extend the sampling theorem to not necessarily bandlimited functions. Note that those functions $f\in L(\mathbf{R})$ that are either bandlimited or duration-limited form a dense subspace of $L(\mathbf{R})$.

This leads to the following theorem, considered by Brown [7], Boas [5], and Butzer-Splettstösser [19].

Theorem 2. If $f\in C(\mathbf{R})\cap L(\mathbf{R})$ and $f^\wedge\in L(\mathbf{R})$, then, uniformly in $t\in\mathbf{R}$,

$$f(t)=\lim_{W\to\infty}\sum_{k=-\infty}^{\infty}f\left(\frac{k}{W}\right)\frac{\sin\pi(Wt-k)}{\pi(Wt-k)}. \tag{2.1}$$

It states that $f(t)$ can be reconstructed from its sampling sum provided one takes its limit as $W\to\infty$. Here the distance between the nodes k/W is not fixed as in Theorem 1 but decreases for $W\to\infty$, so that the number of nodes increases. Thus the series in (2.1) approximates and simultaneously interpolates $f(t)$ at $t=k/W$ for each fixed w.

Let me prove Theorems 1 and 2 by means of a slight modification of the proof of Boas [5], using results which directly precede the basic Poisson summation formula (cf. [18, pp. 123 f, 201 ff.]). Let f, $f^\wedge\in L(\mathbf{R})$. These state that if

$$F^*(v):=\sqrt{W}\sum_{k=-\infty}^{\infty}f^\wedge(2k\pi W-v), \tag{2.2}$$

then this series is dominatedly convergent on every compact interval, and so

$$F^*\in L_{2\pi W};\quad [F^*]^\wedge_C(k)=[f^\wedge]^\wedge(-k/W)=f(k/W)\qquad (k\in\mathbf{R}),$$

where $[g]^\wedge_C(k):=(1/\sqrt{2\pi W})\int_{-\pi W}^{\pi W}g(u)\exp(-iku/W)du$, $k\in\mathbf{R}$, denote the Fourier coefficients of $g\in L_{2\pi W}$ (= class of functions which are $2\pi W$-periodic and Lebesgue integrable over $(-\pi W,\pi W)$). So one has the Fourier expansion

$$F^*(v)\sim\frac{1}{\sqrt{2\pi W}}\sum_{k=-\infty}^{\infty}f\left(\frac{k}{W}\right)e^{ikv/W}\qquad (v\in\mathbf{R}). \tag{2.3}$$

Since a Fourier series can be multiplied by e^{-ivt}, a function of bounded variation, and integrated term by term (see [85, p. 160]), and

$$\frac{1}{2\pi W}\int_{-\pi W}^{\pi W}e^{-itv}e^{i(k/W)v}dv=\mathrm{si}\{\pi(Wt-k)\}\qquad (k\in\mathbf{R}),$$

it follows that (2.3) yields for each $t\in\mathbf{R}$,

$$\frac{1}{\sqrt{2\pi W}}\int_{-\pi W}^{\pi W}F^*(v)e^{-ivt}dv=\sum_{k=-\infty}^{\infty}f\left(\frac{k}{W}\right)\mathrm{si}\{\pi(Wt-k)\}=:S(t).$$

Replacing the function F^* in the integral on the left by the series (2.2), interchanging integral and sum (possible by the dominated convergence), and substi tuting $2k\pi W-v$ by v, then

$$S(t)=\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}e^{-i2k\pi Wt}\int_{(2k-1)\pi W}^{(2k+1)\pi W}f^{\wedge}(v)e^{ivt}\,dv.$$

on the other band, one has

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f^{\wedge}(v)e^{ivt}\,dv=\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty}\int_{(2k-1)\pi W}^{(2k+1)\pi W}f^{\wedge}(v)e^{ivt}\,dv.$$

Thin gives for each $t\in\mathbf{R}$

$$(2.4)\qquad |(R_Wf)(t)|:=|f(t)-S(t)|=\frac{1}{\sqrt{2\pi}}\left|\sum_{k=-\infty}^{\infty}(1-e^{-i2k\pi Wt})\int_{(2k-1)\pi W}^{(2k+1)\pi W}f^{\wedge}(v)e^{ivt}dv\right|$$

$$\leqslant\frac{2}{\sqrt{2\pi}}\sum_{k\neq 0}\int_{(2k-1)\pi W}^{(2k+1)\pi W}|f^{\wedge}(v)|\,dv=\frac{2}{\sqrt{2\pi}}\int_{|v|>\pi W}|f^{\wedge}(v)|\,dv.$$

Now, if $f^{\wedge}$ vanishes outside $[-\pi W,\pi W]$, then $(R_Wf)(t)=0$ for all $t\in\mathbf{R}$, proving Theorem 1. In general, if $f^{\wedge}\in L(\mathbf{R})$, then $\lim_{W\to\infty}(R_Wf)(t)=0$ uniformly in $t\in\mathbf{R}$, establishing Theorem 2.

The above estimate is, as Brown [7] shows, the best result of its kind. There exist different proofs of the sampling theorem: there are those based upon Fourier series expansions, see Brown [7], Butzer-Splettstösser [19,20], upon Parseval's formula, see Brown [8], Stens [74], upon function theory methods (residue formula), see e. g. G. Wunsch [82], upon the Euler-MacLaurin summation formula, see Butzer-Stens [24].

It is possible to weaken the hypotheses of Theorem 2 slightly.

Theorem 2* If $f(t)=(1/\sqrt{2\pi})\int_R g(v)e^{itv}\,dv$, $t\in\mathbf{R}$, *with* $g\in L(\mathbf{R})$, *then*(2.1) *holds uniformly in* $t\in\mathbf{R}$.

Indeed, if f, $f^{\wedge}\in L(\mathbf{R})$, then $g(v)=f^{\wedge}(v)$, $v\in\mathbf{R}$.

3. Sampling Theorem for Duration-Limited Functions; Comparison with Classical Results

If a function f is to be determined by its sampled values $f(k/W)$, $k\in\mathbf{Z}$ in case it is duration-limited, $2N+1$ such values must be evaluated; here $N=N(T,W):=[TW]$ is the largest integer equal to or less than TW. Indeed,

Theorem 3 *Let* $f\in C(\mathbf{R})$ *such that* $f(t)=0$ *for all* $|t|>T$ *and* $f^{\wedge}\in L(\mathbf{R})$. *Then, uniformly in* $t\in\mathbf{R}$,

$$(3.1)\qquad f(t)=\lim_{W\to\infty}\sum_{k=-\infty}^{N}f\left(\frac{k}{W}\right)\frac{\sin\pi(Wt-k)}{\pi(Wt-k)}.$$

The result follows from Theorem 2 by noting that $f(\pm k/W)$ vanishes for $|k|>N$, but does not for $|k|\leqslant N$.

Let us reformulate Theorem 3 so that it is comparable with well-known results on interpolation. Setting $C_{2\pi}(\mathbf{R}):=\{f\in C(\mathbf{R});\ f(t)=0$ for $t\notin[0,2\pi]\}$, an application of Theorem 2 with $W=(2n+1)/2\pi$ gives, together with

$$(3.2)\qquad (T_nf)(t):=\frac{2}{2n+1}\sum_{k=0}^{2n}f\left(\frac{2\pi}{2n+1}k\right)\frac{\sin\frac{(2n+1)}{2}\left(t-\frac{2\pi}{2n+1}k\right)}{\left(t-\frac{2\pi}{2n+1}k\right)}\qquad (n\in\mathbf{N})$$

Corollary 1 Let $f\in C_{2\pi}(\mathbf{R})$ such that $f^\wedge\in L(\mathbf{R})$. Then, uniformly in $t\in\mathbf{R}$,

$$(3.3)\qquad f(t)=\lim_{n\to\infty}(T_nf)(t).$$

The operators $T_n:C_{2\pi}(\mathbf{R})\to C(\mathbf{R})$, defined by (3.2), are bounded for each fixed $n\in\mathbf{N}$ but not uniformly so since the operator norms divergent. Indeed (cf. [73]),

$$\|T_n\|_{[C_{2\pi}(R),C(R)]}=\sup_{t\in R}\frac{2}{2n+1}\sum_{k=1}^{2n}\frac{\sin\frac{2n+1}{2}\left(t-\frac{2\pi}{2n+1}k\right)}{\left(t-\frac{2\pi}{2n+1}k\right)}\cong\log n.$$

The fact that $(T_nf)(t)$ converges uniformly to $f(t)$ fot $n\to\infty$ is no contradiction to the Banach-Steinhaus theorem since in addition to $f\in C_{2\pi}(\mathbf{R})$ it is assumed that $f^\wedge\in L(\mathbf{R})$. Note that T_n is not a polynomial operator, nor does it define a periodic function.

In comparison, let me recall the trigonometric Lagrange interpolating polynomial of $f\in C_{2\pi}$(=class of functions which are continuous and 2π-periodic on $\mathbf{R}$) at the equidistant nodes $t_k:=(2\pi/(2n+1))k$, $0\leqslant k\leqslant 2n$, $n\in\mathbf{P}$(cf. A. Zygmund [85, II, p. 4 ff]). It can be written as

$$(L_nf)(t)=\frac{2}{2n+1}\sum_{k=0}^{2n}f\left(\frac{2\pi}{2n+1}k\right)\frac{\sin\frac{2n+1}{2}\left(t-\frac{2\pi}{2n+1}k\right)}{2\sin\frac{1}{2}\left(t-\frac{2\pi}{2n+1}k\right)}\qquad (t\in\mathbf{R}).$$

This polynomial interpolates f at $t=t_k$: $(L_nf)(t_k)=f(t_k)$. Moreover, $(L_np_n)(t)=p_n(t)$ for any trigonometric polynomial $p_n\in P_n$(=class of all p_n of degree$\leqslant$n). For each fixed $n\in\mathbf{N}$ L_n is a bounded, linear operator mapping $C_{2\pi}$ onto P_n which is idempotent. So on account of the Harsiladze-Losinsiĭ theorem (cf, [27]), it cannot be expected that $(L_nf)(t)$ converges uniformly to $f(t)$ for every $f\in C_{2\pi}$ unless f satisfies in addition some smoothness condition (such as (4.9); cf. [57, § 5.4]). For L_n one has $\|L_n\|_{[C_{2\pi},P_n]}\cong\log n$.

4. Error Estimates for Non-bandlimited Functions

Our next question is the rate of convergence in Theorem 2, in other words, how good is the approximation of $f(t)$ by the sum in (2.1), namely $(R_Wf)(t)$ of (2.3) – often called the aliasing error –, independence upon smoothness conditions on f. These are primarily given in terms of Lipschitz classes. Such a class of order α, $0<\alpha\leqslant 1$, is defined by

(4.1) $$\mathrm{Lip}_L(\alpha;C):=\{f\in C(\mathbf{R}),\ \sup_{|h|\leq\delta}\|f(\cdot+h)-f(\cdot)\|_C\leq L\delta^\alpha,\ \alpha>0\}.$$

Secondly, f is assumed to have a given rate of decay at infinity, $f(t)=O(|t|^{-\gamma})$ for $|t|\geq t_0$, some $0<\gamma\leq 1$. Since this assumption is trivial for $t\leq t_0$ if $f\in(\mathbf{R})$, it is equivalent to

(4.2) $$|f(t)|\leq M_f|t|^{-\gamma}\qquad (t\neq 0)$$

for some $0<\gamma\leq 1$. Note that if $f\in C_{2\pi}(\mathbf{R})$, then $f\in L(\mathbf{R})$ and $|f(t)|\leq\|f\|_C\,2\pi/|t|$ for $t\neq 0$, i. e., (4.2) is satisfied with $\gamma=1$.

For our theorem in this regard, due basically to Splettstösser [62] and Stens [74], two lemmas, contained implicitly in [70], will be needed.

Lemma 1 *One has for* $q>1$, $1/p+1/q=1$, $W>0$,

$$\sum_{k=-\infty}^{\infty}|\mathrm{si}\{\pi(Wt-k)\}|^q\leq 1+\left(\frac{2}{\pi}\right)^q\frac{q}{q-1}<p.$$

Lemma 2. *Assume that condition* (4.2) *is satisfied for some* $\gamma\in(0,1]$. *For each* $p\geq 2/\gamma$, $W>0$, $V>0$,

$$\Big(\sum_{|k|>V}|f(k/W)|^p\Big)^{1/p}\leq M_f\Big(\sum_{|k|>V}\left|\frac{k}{W}\right|^{-\gamma p}\Big)^{1/p}\leq 2^{1/p}M_fW^\gamma V^{(1-p\gamma)/p}.$$

Theorem 4 *Let* $f\in C(\mathbf{R})\cap L(\mathbf{R})$ *satisfy* (4.2) *for some* $0<\gamma\leq 1$. *If* $f^{(r)}\in\mathrm{Lip}_L(\alpha;C)$ *for* $0<\alpha\leq 1$, $r\in\mathbf{P}$, *then*

(4.3) $$\|(R_Wf)(\cdot)\|_C:=\|f(\cdot)-\sum_{k=-\infty}^{\infty}f\Big(\frac{k}{W}\Big)\mathrm{si}\{\pi(W\cdot-k)\}\|_C\leq M_1(f,r,\alpha,\gamma)W^{-r-\alpha}\log W$$

provided $W\geq\exp\{2/(r+\alpha+\gamma)\}$, *with constant* M_1 *given in* (4.7).

Concerning the proof, first consider the de La Vallèe Poussin means, defined for $f\in C(\mathbf{R})$, $t\in\mathbf{R}$, $\rho>0$ by

(4.4) $$(Vp_\rho f)(t):=\frac{3\rho}{2\pi}\int_R f(t-u)\,\mathrm{si}\Big\{\frac{3}{2}\rho u\Big\}\mathrm{si}\Big\{\frac{1}{2}\rho u\Big\}du.$$

They belong to $C(\mathbf{R})\cap L(\mathbf{R})$, they are known (cf. [70]) to satisfy the property (4.2) with $M_f':=3(M_f+\|f\|_C)$ for some $0<\gamma\leq 1$ and $\rho\geq 1$ provided f does. If $f^{(r)}\in\mathrm{Lip}_L(\alpha;C)$, then $\|f(\cdot)-(Vp_\rho f)(\cdot)\|_C\leq 7L\rho^{-r-\alpha}$ for $\rho\geq 1$. Moreover, they are bandlimited to $[-2\rho,2\rho]$. By Theorem 1 this implies that

(4.5) $$(Vp_\rho f)(t)=\sum_{k=-\infty}^{\infty}(Vp_\rho f)\Big(\frac{k}{W}\Big)\mathrm{si}\{\pi(Wt-k)\}$$

for $\rho=\pi W/2$ and each $t\in\mathbf{R}$. Hence

$$(R_Wf)(t)=f(t)-(Vp_\rho f)(t)+\sum_{k=-\infty}^{\infty}\Big\{(Vp_\rho f)\Big(\frac{k}{W}\Big)-f\Big(\frac{k}{W}\Big)\Big\}\mathrm{si}\{\pi(Wt-k)\}=:I_1(t)+I_2(t).$$

Since $\|I_1(\cdot)\|_C\leq C_1W^{-r-\alpha}$ for $W\geq 1$, where $C_1:=7L(2/\pi)^{r+\alpha}$, to complete the proof it suffices to show that $\|I_2(\cdot)|_C=O((\log W)/W^{r+\alpha})$. Indeed, Hölder's inequality yields

$$(4.6)\qquad |I_2(t)|\leqslant\Big(\sum_{k=-\infty}^{\infty}|\mathrm{si}\{\pi(Wt-k)\}|^q\Big)^{1/q}\Big(\sum_{k=-\infty}^{\infty}\Big|(Vp_\rho f)\Big(\frac{k}{W}\Big)-f\Big(\frac{k}{W}\Big)\Big|^p\Big)^{1/p}.$$

Split up the second sum into those $k\in\mathbf{Z}$ with $|k|\leqslant V$, denoting it by I_2^1, and into those with $|k|>V$, denoting it by I_2^2, where v is to be chosen suitably. Then $I_2^1\leqslant(2V+1)^{1/p}\|I_1(\cdot)\|_C$. Choosing $V:=[W^{1+(r+\alpha)/\gamma}+1]$, then $V\geqslant 2$ for $W\geqslant 1$, and

$$(2V+1)^{1/p}\leqslant\Big(\frac{5}{2}\Big)^{1/p}2^{1/p}\exp\Big\{\frac{1}{P}\Big(\frac{r+\alpha+\gamma}{\gamma}\Big)\log W\Big\}=5^{1/p}e$$

provided $p:=((r+\alpha+\gamma)/\gamma)\log W$. Then $I_2^1\leqslant 5^{1/p}e\ c_1W^{-r-\alpha}$. Furthermore, since f satisfies (4.2), Lemma 2 gives

$$I_2^2\leqslant c_2\Big(\sum_{|k|>V}\Big|\frac{k}{W}\Big|^{-\gamma p}\Big)^{1/p}\leqslant c_2 2^{1/p}W^{\gamma}V^{1/p}V^{-\gamma}\leqslant c_2 2^{1/p}2^{1/p}eW^{-r-\alpha}$$

with $c_2:=4M_f+3\|f\|_C$, provided $p\gamma\geqslant 2$. The latter condition is equivalent to $\log W\geqslant 2/(r+\alpha+\gamma)$ or $W\geqslant\exp\ (2/(r+\alpha+\gamma))$.

Since the first sum in (4.6) is bounded by $p^{1/q}<p$ by Lemma 1, a combination of all estimates delivers

$$(4.7)\qquad \|(R_Wf)\|_C\leqslant\frac{c_1}{W^{r+\alpha}}+\Big(\frac{r+\alpha+\gamma}{\gamma}\Big)\{5^{\gamma/2}ec_1+2^{\gamma}c_2e\}\frac{\log W}{W^{r+\alpha}}$$

$$\leqslant\frac{(r+\alpha+\gamma)}{2\gamma}\{\ \gamma c_1+25^{\gamma/2}ec_1+2^{\gamma+1}ec_2\ \}\frac{\log W}{W^{r+\alpha}}$$

provided $W\geqslant\exp(2/(r+\alpha+\gamma))$. This proves the theorem.

Note that the concrete estimate in (4.7) of M_t delivered by the above proof is the sharpest known so far. The hypothesis $f\in L(\mathbf{R})$ was used in the proof only to establish the validity of (4.5). But (4.5) is known to hold provided $f\in C(\mathbf{R})$ satisfies (4.2) (see [70]).

Theorem 3 or Corollary 1 may also be supplied with rates. Indeed, Theorem 4 yields

Corollary 2 *If $f\in C_{2\pi}(\mathbf{R})$ and $f^{(r)}\in\mathrm{Lip}(\alpha;C)$ for $0<\alpha\leqslant 1$, $r\in\mathbf{P}$, then*

$$\Big\|f(\cdot)-\frac{2}{2n+1}\sum_{k=0}^{2n}f\Big(\frac{2\pi}{2n+1}k\Big)\frac{\sin\frac{2n+1}{2}(\cdot-\frac{2\pi}{2n+1}k)}{(\cdot-\frac{2\pi}{2n+1}k)}\Big\|_C=O\Big(\frac{\log n}{n^{r+\alpha}}\Big)\quad(n\to\infty).$$

A weaker form of this corollary, namely under the additional hypothesis that the transform $f^\wedge$ belongs to $L(\mathbf{R})$, and the weaker order $O(n^{-r-\alpha-1})$, was established earlier by Butzer-Splettstösser [19],[20]. Moreover, Corollary 2 also holds for $r=0$, thus for functions which need not be differentiable at all. The estimate in Corollary 2 may also be stated in terms of the rth modulus of continuity; see

Butzer [12] where the proof follows as an application of the Banach-Steinhaus theorem with rates.

As an example, consider the function (see [70]),

$$f(t):=\begin{cases}\sin t, & \text{for } t\in[0,2\pi]\\ 0, & \text{for } t \text{ otherwise.}\end{cases}$$

Noting that $f\in \mathrm{Lip}(1;C)$, an application of Corollary 2 yields for $n\to\infty$

$$\left\|f(\cdot)-\frac{2}{2n+1}\sum_{k=0}^{2n}\sin\left(\frac{2\pi}{2n+1}k\right)\frac{\sin\frac{2n+1}{2}\left(\cdot-\frac{2\pi}{2n+1}k\right)}{\left(\cdot-\frac{2\pi}{2n+1}k\right)}\right\|_C=O\left(\frac{\log n}{n}\right).$$

If one would instead take the function

$$g(t):=\int_0^t f(u)du=\begin{cases}1-\cos t, & t\in[0,2\pi]\\ 0, & t \text{ otherwise,}\end{cases}$$

then the error would be of order $O((\log n/n^2)$.

It is of interest to compare the assertions of Corollaries 1 and 2 with the corresponding ones in the case of the partial sums of the Fourier series of $f\in C_{2\pi}$, defined by

$$(4.8)\qquad (S_nf)(t):=\frac{1}{2\pi}\int_0^{2\pi}f(u)\frac{\sin\frac{2n+1}{2}(t-u)}{\sin\frac{1}{2}(t-u)}du\qquad (t\in\mathbf{R}).$$

In this regard, if $f\in C_{2\pi}$ satisfies the Dini-Lipschitz condition, i. e.,

$$(4.9)\qquad \omega(f;C_{2\pi}):=\sup_{|h|\leqslant\delta}\|f(\cdot+h)-f(\cdot)\|_{C_{2\pi}}=o(|1/\log\delta|)\qquad(\delta\to0+),$$

then it is known that (cf. [18, p. 105]) $\lim_{n\to\infty}(S_nf)(t)=f(t)$ uniformly in $t\in\mathbf{R}$. Whereas Corollary 1 is a rough counterpart to this result, an "exact" one also holds. Indeed, R. Stens [73] improved Corollary 1 (actually the proof of Corollary 2) to the effect that $f\in C_{2\pi}(\mathbf{R})$ together with

$$\sup_{|h|\leqslant\delta}\|f(\cdot+h)-f(\cdot)\|_C=o(|1/\log\delta|)\qquad(\delta\to0+)$$

implies the validity of (3.3).

The exact counterpart of Corollary 2 for Fourier series, actually its model, states that for any $f\in C_{2\pi}$ satisfying $\omega(f^{(r)};C_{2\pi})=O(\delta^\alpha)$, $\delta\to0+$, for $0<\alpha\leqslant1$, $r\in\mathbf{P}$, one has

$$(4.10)\qquad \|f(\cdot)-(S_nf)(\cdot)\|_{C_{2\pi}}=O\left(\frac{\log n}{n^{r+\alpha}}\right)\qquad(n\to\infty).$$

One could also compare Theorem 4 with the corresponding one for Fourier integrals. In this respect, Splettstösser [66] has just shown that

$$\text{(4.11)} \qquad \|f(\cdot)-W\int_R f(u)\frac{\sin\pi W(\cdot-u)}{\pi W(\cdot-u)}du\|_C=O\left(\frac{\log W}{W^{r+\alpha}}\right),$$

for $W\to\infty$ provided $f\in C(\mathbf{R})$ satisfies (4.2) for $0<\gamma\leqslant 1$ and $f^{(r)}\in Lip(\alpha;\ C)$. Note that one may regard the infinite sum in (4.3) as a discrete form of the convolution integral in (4.11). If $f\in L(\mathbf{R})$ is bandlimited to $[-\pi W,\pi W]$, then Parseval's formula states in this regard that

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\pi W}^{\pi W}f^\wedge(v)e^{ivt}dv=W\int_R f(u)\frac{\sin\pi W(t-n)}{\pi W(t-u)}du.$$

So (4.11) may be considered as a continuous counterpart of the (discrete) Theorem 4, both being valid for not necessarily bandlimited functions.

5. A Comparison of Fourier Analysis on the Spaces $L^2(-\pi W,\pi W)$, $L^2(\mathbf{R})$ with that on the Paley-Wiener Space $B_{\pi W}$.

5.1 The Sampling Series as an Orthogonal Expansion in the Hilbert Space $B_{\pi W}$.

The cardinal series has properties that are comparable to those of Fourier series as well as of Fourier integrals, as was noted in Sections 3 and 4. In this line let us first consider the matter from the point of view of orthogonal series in a Hilbert space. The sequence $\{\varphi_k(u)\}_{k\in Z}$, where $\varphi_k(u):=(2\pi W)^{-1/2}e^{i(k/W)v}$, is an orthogonal sequence in $L^2(-\pi W,\pi W)$ that is also total. Indeed,

$$\text{(5.1)} \qquad \int_{-\pi W}^{\pi W}\frac{1}{\sqrt{2\pi W}}e^{-ivu}\frac{1}{\sqrt{2\pi W}}e^{+i(j/W)u}du$$
$$=\sqrt{2\pi W}\,\mathrm{si}\left\{\pi W\left(v-\frac{j}{W}\right)\right\}=\begin{cases}1, & k=j\\ 0, & k\neq j,\end{cases}$$

where $v=k/W$; moreover $\int_{-\pi W}^{\pi W}f(u)\overline{\varphi_k(u)}du=0$ for $f\in L^2(-\pi W,\pi W)$ and all $k\in\mathbf{Z}$ implies that $f\equiv 0$. Considering the space $L^2(\mathbf{R})$ now, the sequence $\{\varphi_{k,+}(u)\}_{k\in Z}$, given by

$$\text{(5.2)} \qquad \varphi_{k,+}(u):=\begin{cases}(2\pi W)^{-1/2}e^{i(k/W)u}, & |u|<\pi W\\ 0, & |u|>\pi W,\end{cases}$$

lies in $L^2(\mathbf{R})$ with $\int_R\varphi_{j,+}(u)\varphi_{k,+}(u)du=\delta_{j,k}$ by (5.1); so it defines an orthonormal sequence in $L^2(\mathbf{R})$. Now by (5.1)

$$[\varphi_{k,+}]^\wedge(v):=L^2(\mathbf{R})\frac{1}{\sqrt{2\pi}}\int_R e^{-ivu}\varphi_{k,+}(u)du=\sqrt{W}\,\mathrm{si}\left\{\pi W\left(v-\frac{k}{W}\right)\right\}\quad(k\in\mathbf{Z};\ v\in\mathbf{R})$$

is also an orthonormal sequence in $L^2(\mathbf{R})$ since

$$\int_R[\varphi_{j,+}]^\wedge(v)[\varphi_{k,+}]^\wedge(v)\,dv=\frac{1}{2\pi W}\int_{-\pi W}^{\pi W}e^{+i(j/W)v}e^{-i(k/W)v}dv=\delta_{j,k}$$

by Parseval's formula (cf. [18, p. 193]).

Since of course the sequence $\{\varphi_{k,+}(u)\}_{k\in Z}$ is not total in $L^2(\mathbf{R})$, it is not to be expected that $\{[\varphi_{k,+}]^\wedge(t)\}_{k\in Z}$ is total in $L^2(\mathbf{R})$. However, the latter sequence is total in the subspace $B_{\pi W}$ of $L^2(\mathbf{R})$. Indeed, these functions belong to $B_{\pi W}$ since

$$|[\varphi_{k,+}]^\wedge(z)|\leqslant\sqrt{W}e\, e^{\pi W|y|} \qquad (k\in\mathbf{Z};\ z=t+iy).$$

Moreover, for $g\in B_{\pi W}$,

$$\int_R g(t)\overline{[\varphi_{k,+}]^\wedge(t)}dt=\int_R g^\wedge(v)\varphi_{k,+}(v)dv=\int_{-\pi W}^{\pi W}g^\wedge(v)\varphi_k(v)dv \tag{5.3}$$

by Parseval's theorem, valid for f, $g\in L^2(\mathbf{R})$,

$$\int_R g(t)f^\wedge(t)dt=\int_R g^\wedge(v)f(v)dv. \tag{5.4}$$

Now if the integral on the left side of (5.3) vanishes for all $k\in\mathbf{Z}$, the totality of φ_k in $L^2(-\pi W,\pi W)$ implies that $g^\wedge(v)$ is null on $[-\pi W,\pi W]$, so that $g(t)=1/\sqrt{2\pi}\int_{-\pi W}^{\pi W}e^{itv}g^\wedge(v)dv=0$ by (1.2) for all $t\in\mathbf{R}$.

So it is of interest to consider the sequence $\{[\varphi_{k,+}]^\wedge(t)\}_{k\in Z}=\left\{\sqrt{W}\mathrm{si}\,W\left(t-\frac{k}{W}\right)\right\}_{k\in Z}$ in the class $B_{\pi W}$. But $B_{\pi W}$ is a vector space under pointwise addition and scalar multiplication; it is an inner product space with respect to $(f,g):=\int_R f(t)\cdot\overline{g(t)}\,dt$. Since the Fourier transform is an isometry, the Paley-Wiener theorem shows that $B_{\pi W}$ is a separable Hilbert space, isometrically isomorphic to $L^2(-\pi W,\pi W)$. G. H. Hardy [35] actually called the functions of $B_{\pi W}$ the Paley-Wiener functions, so that $B_{\pi W}$ may be called the Paley-Wiener space.

[Note that the fact that the Fourier transform is an isometric mapping, here of the total orthonormal sequence $\{\varphi_{k,+}\}_{k\in Z}\subset L^2(\mathbf{R})$ into $\{[\varphi_{k,+}]^\wedge\}_{k\in Z}\subset B_{\pi W}$, could also have been used to prove that the sequence $\{[\varphi_{k,+}]^\wedge\}_{k\in Z}$ is total and orthonormal by a general result; see Higgins [36, p. 55]].

Since the sequence $\{\sqrt{W}\mathrm{si}\,\pi(Wt-k)\}_{k\in Z}$ forms a total orthonormal sequence, thus a basis in $B_{\pi W}$, it is possible to apply the welldeveloped theory of (general) orthogonal sequences in Hilbert spaces to this particular case. First of all, for any $f\in B_{\pi W}$

$$\left\|f(t)-\sum_{k=-n}^{n}\gamma_k\sqrt{W}\mathrm{si}\{\pi(Wt-k)\}\right\|_{L^2(R)} \tag{5.5}$$

takes on its minimal value when the γ_k are the Fourier coefficients of f with respect to the sequence $\{\sqrt{W}\mathrm{si}(\pi Wt-k)\}_{k\in Z}$, i. e.,

$$\gamma_k=(f,[\varphi_{k,+}]^\wedge)=\frac{1}{\sqrt{2W\pi}}\int_{-\pi}^{\pi W}f^\wedge(v)e^{i(k/W)v}dv=\frac{1}{\sqrt{W}}f\left(\frac{k}{W}\right) \tag{5.6}$$

in view of (5.4) and (1.2). This means that

$$\sum_{k=-n}^{n}f\left(\frac{k}{W}\right)\mathrm{si}\{\pi W(t-k)\} \tag{5.7}$$

gives the best approximation to f by "polynomials" $\sum_{k=-n}^{n}\gamma_k\sqrt{W}\mathrm{si}\{\pi(Wt-k)\}$ in the metric of $B_{\pi W}$.

Moreover, the series (5.7) converges in $B_{\pi W}$ to f, giving

$$(5.8)\qquad f(t)=\sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right)\mathrm{si}\{\pi(Wt-k)\}$$

in $L^2(\mathbf{R})$-norm. Furthermore, Parseval's equation for $f\in B_{\pi W}$ together with (5.6) reads

$$(5.9)\qquad \int_R |f(t)|^2dt=\sum_{k=-\infty}^{\infty}|f,[\varphi_{k,+}]^\wedge)|^2=\frac{1}{W}\sum_{k=-\infty}^{\infty}\left|f\left(\frac{k}{W}\right)\right|^2.$$

All in all, each $f\in B_{\pi W}$ has the unique Fourier expansion (5.8), so that the Fourier series of $f\in B_{\pi W}$ with respect to the sequence $\{[\varphi_{k,+}]^\wedge\}_{k\in Z}$ actually turns out to be the cardinal series of f.

Finally, the convergence in $B_{\pi W}$ implies pointwise and uniform convergence over $\mathbf{R}$. Indeed, choosing $f(u):=\mathrm{si}\{\pi(Wt-k)\}$ in (5.9) yields

$$\frac{1}{W}\sum_{k=-\infty}^{\infty}|\mathrm{si}\,\{\pi(Wt-k)\}|^2=\int_R\left[\frac{\sin\pi(Wt-u)}{\pi(Wt-u)}\right]^2du=1.$$

By Hölder's inequality this gives

$$\sum_{|k|>n}|f\left(\frac{k}{W}\right)\mathrm{si}\{\pi(Wt-k)\}|\leqslant\left[\sum_{|k|>n}f\left|\left(\frac{k}{W}\right)\right|^2\right]^{1/2}\left[\sum_{k=-\infty}^{\infty}|\mathrm{si}\pi\{(Wt-k)\}|^2\right]^{1/2}$$

$$=\left[W\sum_{|k|>n}\left|f\left(\frac{k}{W}\right)\right|^2\right]^{1/2}$$

which tends to zero uniformly in $t\in\mathbf{R}$ for $n\to\infty$ by (5.9). Hence the series (5.7) tends in $L^2(\mathbf{R})$-norm towards a function $g\in L^2(\mathbf{R})$. But since the $L^2(\mathbf{R})$-limit of (5.7) is f, one has $g\equiv f$.

On the other hand, by the Riesz-Fischer theorem there exists an $f\in B_{\pi W}$ such that (5.5) converges to zero for $n\to\infty$ if and only if $\sum_{-\infty}^{\infty}|\gamma_k|^2<\infty$. In this case $\gamma_k=(f,[\varphi_{k,+}]^\wedge)$. Note that f is unique since the sequence $\{[\varphi_{k,+}]^\wedge\}_{k\in Z}$ is total.

For this material see also [36, pp. 57f.],[83, pp. 105ff.],[48] and [72].

5.2 The Special Role of the Space $B_{\pi W}$

The cardinal function has been described as "a function of royal blood in the family of entire functions, whose distinguished properties separate it from its bourgeois brethren" (see also [48]). The Paley-Wiener theorem reconfirmed that the space $B_{\pi W}$ does indeed play a special role in analysis. Just as the Poisson summation formula provides a basic link between Fourier series and Fourier integrals, which is by no means trivial in the case of not necessarily bandlimited functions, so does the classical sampling theorem in the bandlimited case.

In this line of thought let us consider the material presented so far in retrospect. Let us set $W=1$ for simplicity. First of all, the space B_π "lies between" $L^2(-\pi,\pi)$ and $L^2(\mathbf{R})$. Indeed,

$$L^2_{2\pi} \cong L^2(-\pi,\pi) = \{g \in L^2(\mathbf{R});\ g(t)=0,\ |t|>\pi\} \cong B_\pi \subset L^2(\mathbf{R}).$$

(The fact that $L^2(-\pi,\pi)$ is isomorphic to B_π results from the Paley-Wiener theorem). Furthermore, the space B_π has the best properties of both the spaces $L^2(-\pi,\pi)$ and $L^2(\mathbf{R})$. This can be seen from the table, where the Fourier analysis on the three spaces is set up parallel to another. Here $L^2(\mathbf{R})-\int_R$ stands for the Fourier transform for $L^2(\mathbf{R})$-functions, $L^2(\mathbf{R})-\Sigma_{k\in Z}$ denotes the limit for $n\to\infty$ of $\Sigma^n_{k=-n}$ in $L^2(\mathbf{R})$-norm, and l^2 is the space of all complex sequences $\{\xi_k\}_{k\in Z}$ with $\Sigma_{k\in Z}|\xi_k|^2<+\infty$.

Spaces	$L^2(-\pi,\pi)$	B_π	$L^2(\mathbf{R})$
Total, Orthonormal Sequence	$\{(2\pi)^{-1/2}e^{ikt}\}_{k\in Z}$	$\{\mathrm{si}\,\pi(t-k)\}_{k\in Z}$	
Orthonormality	$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt}e^{-ijt}dt = \delta_{j,k}$	$\int_R \mathrm{si}(\pi(t-k))\mathrm{si}(\pi(t-j))dt = \delta_{j,k}$	
Fourier Transforms	$f_C^\wedge(k) := \frac{1}{\sqrt{2\pi}} \cdot \int_{-\pi}^{\pi} f(u)e^{-iku}du$ $(k\in\mathbf{Z})$ $f_C^\wedge: L^2(-\pi,\pi)\to l^2$	$f_B^\wedge(k) := (f, \mathrm{si}\,\pi(\cdot-k)) = f(k)$ $(k\in\mathbf{Z})$ $f_B^\wedge: B_\pi\to l^2$	$f^\wedge(v) := L^2(\mathbf{R})-\frac{1}{\sqrt{2\pi}}\int_R f(u)e^{-ivu}du$ $(v\in\mathbf{R})$ $f^\wedge: L^2(\mathbf{R})\to L^2(\mathbf{R})$ (5.10)
Inversion Formulae	$f(t) = L^2(\mathbf{R}) - \frac{1}{\sqrt{2\pi}}\sum_{k\in Z} f_C^\wedge(k)e^{ikt}$	$f(t) = L^2(\mathbf{R}) - \sum_{k\in Z} f_B^\wedge(k)\mathrm{si}(\pi(t-k))$	$f(t) = L^2(\mathbf{R}) - \frac{1}{\sqrt{2\pi}} \times \int_R f^\wedge(v)e^{itv}dv$ (5.11)
n-th Partial Sums (Integrals)	$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\times \frac{\sin(2n+1)(t-u)/2}{\sin(t-u)/2} \times du$	$\int_{-\pi}^{\pi} f(u)D_n(u,t)du$ $D_n(u,t) := \sum_{k=-n}^{n}\mathrm{si}(\pi(t-k))\mathrm{si}(\pi(u-k))$	$\frac{1}{\sqrt{2\pi}}\int_{-n}^{n} f^\wedge(v)e^{itv}dv = \frac{1}{\pi}\int_R f(u) \times\frac{\sin(n(t-u))}{(t-u)}du$ (5.12)
Generalized Parseval Equations	$\sum_{k\in Z} f_C^\wedge(k)\overline{g_C^\wedge(k)} = \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt$ $(f,g\in L^2(-\pi,\pi))$	$\sum_{k\in Z} f_B^\wedge(k)\overline{g_B^\wedge(k)} = \int_R f(t)\overline{g(t)}dt$ $(f,g\in B_\pi)$	$\int_R f^\wedge(v)\overline{g^\wedge(v)}dv = \int_R f(t)\overline{g(t)}dt$ $(f,g\in L^2(\mathbf{R}))$.

TABLE

Note that whereas the convolution product for the spaces $L^2(-\pi,\pi)$ and $L^2(\mathbf{R})$ is defined by $\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(t-u)g(u)du$ and $(1/\sqrt{2\pi})\int_R f(t-u)g(u)du$, respectively, the convolution of $f,g\in B_\pi$ takes on the simple form $(f*g)(t):=f(t)\cdot g(t)$ for $t\in\mathbf{R}$ provided $f\cdot g\in B_\pi$, with the convolution theorem $[f*g]_B^\wedge(k)=f^\wedge(k)g_B^\wedge(k)$, $k\in\mathbf{Z}$.

Let us finally compare transform theory for the three spaces $L^2(-\pi,\pi)$, B_π and $L^2(\mathbf{R})$, more specifically between the latter two. Indeed, if $f\in B_\pi$, then $f^\wedge\in L^2(-\pi,\pi)$, so $f^\wedge$ can be expanded into its (classical) Fourier series (cf. [18, p. 175])

$$
\begin{aligned}
f^\wedge(v) &= L^2(\mathbf{R})-\sum_{k=-\infty}^{\infty}[f^\wedge]_C^\wedge(k)\left\{\frac{1}{\sqrt{2\pi}}e^{ikv}\right\}_+ \\
&= L^2(\mathbf{R})-\sum_{k=-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f^\wedge(u)e^{-iku}du\right)\left\{\frac{1}{\sqrt{2\pi}}e^{iku}\right\}_+ \qquad (5.13)\\
&= L^2(\mathbf{R})-\frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{\infty} f(k)\{e^{-iku}\}_+.
\end{aligned}
$$

In other words, the restriction of the Fourier transform (5.10) for $L^2(\mathbf{R})$-functions to B_π-functions yields the transform (5.13); it is actually a discretized form of (5.10). On the other hand, the $L^2(\mathbf{R})$ inverse transform (5.11) restricted to $f\in B_\pi$ turns out to be, noting (5.13),

$$
f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f^\wedge(v)e^{ivt}dv = L^2(\mathbf{R})-\sum_{k=-\infty}^{\infty} f(k)\,\mathrm{si}\{\pi(t-k)\}
$$

in view of (5.1). This is the Fourier series expansion of f in terms of the sequence $\{\mathrm{si}\pi(t-k)\}_{k\in Z}$ in the $L^2(\mathbf{R})$-metric, and can be regarded as the discretized version of the Fourier inversion integral in the form (5.12).

6. Round-off Error in Sampling Series

When setting up the sampling sum it may happen that one does not have the exact sample values $f(k/W)$ at one's disposal but only recorded or tabulated values $\bar f(k/W)$, both differing by

$$
\varepsilon_k := f(k/W)-\bar f(k/W)
$$

where $|\varepsilon_k|\leqslant\varepsilon$, $k\in\mathbf{Z}$; ε_k is called the local round-off error. In digital signal processing this is the case when the sampled values are replaced by the nearest discrete (quantized) values, namely by the values $\bar f(k/W)$ of the corresponding step function $\bar f$ with possible values $2r\varepsilon$, $r\in\mathbf{Z}$ (see Figure). This is assumed below, then $|\varepsilon_k|\leqslant|f(k/W)|$, $k\in\mathbf{Z}$.

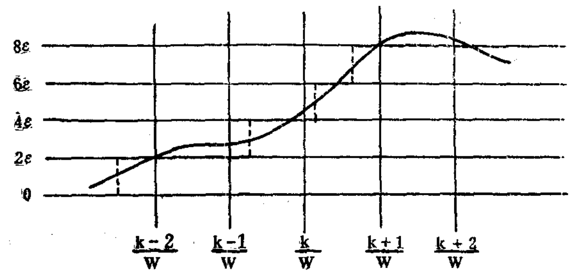

Figure Function f (drawn as———) and its corresponding step function $\bar{f}$(………) with values $f(k/W)$ and $\bar{f}(k/W)$, respectively.

In this regard it is of interest to consider the total round-off (or quantization) error

$$(Q_\varepsilon f):=(Q_{\varepsilon,W}f)(t):=f(t)-\sum_{k=-\infty}^{\infty}\bar{f}\left(\frac{k}{W}\right)\mathrm{si}\{\pi(Wt-k)\}=\sum_{k=-\infty}^{\infty}\varepsilon_k\,\mathrm{si}\{\pi(Wt-k)\}\quad(t\in\mathbf{R})$$

under the hypotheses of Theorem 1. Round-off errors, possibly caused by uncertaincies in the sample values, are generally treated using stochastic methods; see e. g. D. S. Ruchkin [52], A. Papoulis [49] and T. Ericson [31]. Our main result here, which employs deterministic methods, states

Theorem 5 *a) Let $f\in C(\mathbf{R})\cap L(\mathbf{R})$ with $f^\wedge(v)=0$ for all $|v|>\pi W$, $W>0$. Then, uniformly in $t\in\mathbf{R}$,*

$$\lim_{\varepsilon\to 0+}(Q_\varepsilon f)(t)=0.$$

b) In addition assume that (4.2) holds for some $\gamma\in(0,1]$. Then

$$\|(Q_\varepsilon f)(\cdot)\|_C\leqslant M_2(f,\gamma)\varepsilon\log(1/\varepsilon)$$

for $|\varepsilon_k|\leqslant\varepsilon\leqslant\min\{1/W,\ e^{-1/2}\}$, $k\in\mathbf{Z}$, $W\geqslant 1$, where M_2 is the constant of (6.3).

Concerning the proof of part a), Hölder's inequality gives

$$|(Q_\varepsilon f)(t)|\leqslant\left(\sum_{k=-\infty}^{\infty}|\mathrm{si}\{\pi(Wt-k)\}|^q\right)^{1/q}\cdot\left(\sum_{k=-\infty}^{\infty}|\varepsilon_k|^p\right)^{1/p}.\tag{6.1}$$

Now the first sum is bounded for $q=2$ by 2 according to Lemma 1. Regarding the second sum in (6.1), since $|\varepsilon_k|\leqslant|f(k/W)|$, $k\in\mathbf{Z}$,

$$\left(\sum_{k=-\infty}^{\infty}|\varepsilon_k|^2\right)^{1/2}\leqslant\left(\sum_{|k|\leqslant[1/\varepsilon]}|\varepsilon_k|^2\right)^{1/2}+\left(\sum_{|k|>[1/\varepsilon]}\left|f\left(\frac{k}{W}\right)\right|^2\right)^{1/2}.$$

The first term on the right side has upper bound $\sqrt{(2[1/\varepsilon]+1)\varepsilon^2}\leqslant\sqrt{2\varepsilon+\varepsilon^2}$, which tends to zero for $\varepsilon\to 0+$. The second term also tends to zero in view of the convergence of the series

$$\sum_{k=-\infty}^{\infty}\left|f\left(\frac{k}{W}\right)\right|^2=\int_{-\pi W}^{\pi W}|f^\wedge(v)|^2dv,$$

valid in view of Parseval's formula since $f^\wedge \in L^2(-\pi W, \pi W)$.

Concerning part b), the second sum in (6.1) is bounded by

$$(6.2) \qquad \Big(\sum_{|k|<a(p,\varepsilon)} |\varepsilon_k|^p\Big)^{1/p} + \Big(\sum_{|k|>a(p,\varepsilon)} \Big|f\Big(\frac{k}{W}\Big)\Big|^p\Big)^{1/p},$$

where $a(p,\varepsilon) := [\varepsilon^{-1/\gamma} W^{p\gamma/(p\gamma-1)}]$. Firstly, note that $a(p,\varepsilon) \geqslant 1$ provided $\varepsilon \leqslant 1/W$, $W \geqslant 1$, and $p\gamma \geqslant 2$. Then the first term in (6.2), noting that $W^\gamma < 1/\varepsilon$ for $\gamma \in (0,1]$, is bounded by

$$(\{2a(p,\varepsilon)+1\}\varepsilon^p)^{1/p} \leqslant 3^{1/p}\varepsilon\, \varepsilon^{-1/p\gamma} W^{\gamma/(p\gamma-1)} \leqslant 3^{1/p}\varepsilon \exp\{(4/p\gamma)\log(1/\varepsilon)\} \leqslant 3^{\gamma/2}\varepsilon e$$

if one chooses $p = (4/\gamma)\log(1/\varepsilon)$, observing that $1/(p\gamma-1) \leqslant 3/p\gamma$ for $p\gamma \geqslant 2$, and $3^{1/p} \leqslant 3^{\gamma/2}$ for $\varepsilon \leqslant e^{-1/2}$. The second term in (6.2) can then be estimated on account of Lemma 2 by

$$2^{1/p} M_f W^\gamma a(p,\varepsilon)^{(1-p\gamma)/p} \leqslant 2^{\gamma/2} M_f \varepsilon\, e^{1/4}.$$

Combining all the results, observing that the first sum in (6.1) is less than p, one has

$$(6.3) \qquad |(Q_\varepsilon f)(t)| \leqslant \frac{4}{\gamma}(3^{\gamma/2}e + 2^{\gamma/2} M_f e^{1/4})\varepsilon \log\frac{1}{\varepsilon}$$

uniformly in $t \in \mathbf{R}$. Note that $p\gamma \geqslant 2$ as $\varepsilon \leqslant e^{-1/2}$. This proves part b).

Observe that the convergence given in Theorem 5a) holds uniformly in $t \in \mathbf{R}$. That given in the corresponding Theorem 3.1 of [21] holds only for local t-intervals (namely $|t| \leqslant [1/\varepsilon]/(2W)$). The proof of part b), which is quite different from that of Theorem 3.2 of [21], gets along with less restrictive conditions upon ε, Here $\varepsilon_k = O(|k|^{-\gamma})$, $|k| \to \infty$ in view of (4.2).

It is also possible to examine the total round-off or quantization error in the case of sampling approximation of non-bandlimited functions, thus in the case of Theorem 2. In comparison with Theorem 5b) it will now be an error additional to that caused by the non-bandlimitation, namely to that of the aliasing error of Theorem 4.

Theorem 6 *Under the hypotheses of Theorem 4 one has for* $W = (1/\varepsilon)^{1/(r+\alpha)}$, $r \geqslant 1$,

$$\Big\|f(\cdot) - \sum_{k=-\infty}^{\infty} \bar f\Big(\frac{k}{W}\Big)\mathrm{si}\{\pi(W\cdot - k)\}\Big\|_C \leqslant M_3(f,r,\alpha,\gamma)\,\varepsilon\log(1/\varepsilon)$$

for $\varepsilon \leqslant \exp\{-2(r+\alpha)/(r+\alpha+\gamma)\}$, *where* $M_3 := M_1 + M_2$, M_1 *and* M_2 *being the constants in* (4.7) *and* (6.3).

To prove it, one splits up the term within the norm as

$$\Big\{f(t) - \sum_{-\infty}^{\infty} f\Big(\frac{k}{W}\Big)\mathrm{si}\{\pi(Wt-k)\}\Big\} + \sum_{-\infty}^{\infty}\Big(f\Big(\frac{k}{W}\Big) - \bar f\Big(\frac{k}{W}\Big)\Big)\mathrm{si}\{\pi(Wt-k)\}.$$

The norm of the first term is of order $O(\log W/W^{r+\alpha})$ by Theorem 4, that of the second term of order $O(\varepsilon\log 1/\varepsilon)$ by Theorem 5b). Taking W as postulated, the result follows.

Note that it can be shown that Theorem 6 also holds for $r=0$ by modifying the proof of Theorem 5b) slightly. It is not to be expected that the rate of convergence in Theorem 6 can be any better than $O(\varepsilon\log 1/\varepsilon)$, even if higher derivatives of f exist. However, in this situation the sampling rate can be diminished without increasing the error. Indeed, the existence of the $(r+2)$th continuous derivative instead of just the $(r+1)$th allows one to multiply the distance between the sampling nodes by the factor $\exp(-1/(r+r^2))$. For example, for $r=1$ and $\varepsilon=1/4$ this will be the factor 2.

The assertions of Theorems 5 b) and 6 may be interpreted in terms of stability theory with rates; a small change in the function values at all of the nodes produces a corresponding small change in the sampling expansion on the entire $\mathbf{R}$.

Note that Theorem 6 holds in particular for duration limited functions.

7. Time Jitter Error in the Sampling Theorem

When trying to set up the sampling sums it may also happen that the samples cannot be taken at the instants k/W but at $(k/W+\delta_k)$, the sampled values now being $f(k/W+\delta_k)$. These errors in timing give rise to the jitter error

$$(J_\delta f)(t):=(J_{\delta,W}f)(t):=f(t)-\sum_{k=-\infty}^{\infty}f\left(\frac{k}{W}+\delta_k\right)\mathrm{si}\{\pi(Wt-k)\}$$

$$=\sum_{k=-\infty}^{\infty}\left(f\left(\frac{k}{W}\right)-f\left(\frac{k}{W}+\delta_k\right)\right)\mathrm{si}\{\pi(Wt-k)\}\qquad (t\in\mathbf{R})$$

under the hypotheses of Theorem 1.

The calculation of this error has so far been carried out using stochastic methods-the δ_k being regarded as a weak sense stationary discrete-parameter random process having finite variance; see e. g. A. V. Balakrishnan [1], W. M. Brown-C. J. Palermo [10], F. J. Beutler-O. A. Z. Leneman [3], and Beutler [2]. It will here be treated using strictly deterministic methods, based on the sole assumption that $|\delta_k|\leqslant\delta$ for $k\in\mathbf{Z}$, $\delta>0$ sufficiently small. This answers a question on the deterministic nature of jittered sampling also raised by J. R. Higgins [38].

Theorem 7 *Let $f\in C(\mathbf{R})\cap L(\mathbf{R})$ satisfy (4.2) for $0<\gamma\leqslant 1$ with $f^\wedge(v)=0$ for all $|v|>\pi W$, $W>0$. Then*

$$\|(J_\delta f)(\cdot)\|_C\leqslant M_4(f,f',\gamma)\,\delta\log(1/\delta)$$

provided $|\delta_k|\leqslant\delta\leqslant\min\{1/W,e^{-1/2}\}$, $k\in\mathbf{Z}$, $W\geqslant 1$, where M_4 is the constant of (7.3).

Regarding the proof, by Hölder's inequality

$$(7.1)\qquad |(J_\delta f)(t)|\leqslant\Big(\sum_{k=-\infty}^{\infty}|\mathrm{si}\{\pi(Wt-k)\}|^q\Big)^{1/q}\Big(\sum_{k=-\infty}^{\infty}\Big|f\Big(\frac{k}{W}\Big)-f\Big(\frac{k}{W}+\delta_k\Big)\Big|^p\Big)^{1/p}$$

for $1/p+1/q=1$. The second sum is bounded by

$$(7.2)\qquad (2b+3)^{1/p}\sup_{k\in Z}\|f(\cdot)-f(\cdot+\delta_k)\|_C+\Big(\sum_{|k|>b+2}\Big|f\Big(\frac{k}{W}\Big)-f\Big(\frac{k}{W}+\delta_k\Big)\Big|^p\Big)^{1/p},$$

where $b=b(p,\delta):=[\delta^{-1/\gamma}W^{p\gamma/(p\gamma-1)}]$. Again $b\geqslant 1$ provided $\delta\leqslant 1/W$, $W\geqslant 1$ and $p\gamma\geqslant 2$. The first term in (7.2) is bounded by $(2b+3)^{1/p}\delta\|f'\|_C$ noting that $f'\in C(\mathbf{R})$ by the hypotheses. Then, since $W^\gamma<1/\delta$ for $0<\gamma\leqslant 1$,

$$(2b+3)^{1/p}\leqslant(5b)^{1/p}\leqslant 5^{1/p}\exp\{(4/p\gamma)\log(1/\delta)\}\leqslant 5^{\gamma/2}e$$

if one chooses $p=(4/\gamma)\log(1/\delta)$ and takes $\delta\leqslant e^{-1/2}$. The second term in (7.2) is now bounded by Lemma 2 by

$$2\cdot 2^{1/p}M_fW^\gamma b^{(1-p\gamma)/p\gamma}\leqslant 2\cdot 2^{\gamma/2}M_f\delta e^{1/4}.$$

Since the first sum in (7.1) is bounded by p, a combination of all estimates yields

$$(7.3)\qquad |(J_\delta f)(t)|\leqslant\frac{4}{\gamma}\{5^{\gamma/2}e\|f'\|_C+2\cdot 2^{\gamma/2}M_fe^{1/4}\}\delta\log\Big(\frac{1}{\delta}\Big).$$

The foregoing theorem seems to be new. Deterministic methods have previously been used in Butzer-Splettstösser [22] to study the jitter error for generalized sampling sums which are discretizations of convolution integrals on $\mathbf{R}$.

It is also feasible to study the jitter error in the case of not necessarily band-limited functions.

Theorem 8 *Under the hypotheses of Theorem* 4 *one has for* $W=(1/\delta)^{1/(r+\alpha)}$, $r\geqslant 1$,

$$\|f(\cdot)-\sum_{k=-\infty}^{\infty}f\Big(\frac{k}{W}+\delta_k\Big)\mathrm{si}\{\pi(W\cdot-k)\}\|_C\leqslant M_5(f,f',r,\alpha,\gamma)\ \delta\log(1/\delta)$$

for $\delta\leqslant\exp\{-2(r+\alpha)/(r+\alpha+\gamma)\}$, *where* $M_5:=M_1+M_4$, M_1 *and* M_4 *being the constants in* (4.7) *and* (7.3).

For the proof, split up the term in the norm as

$$\{f(t)-\sum_{k=-\infty}^{\infty}f\Big(\frac{k}{W}\Big)\mathrm{si}\{\pi(Wt-k)\}+\sum_{k=-\infty}^{\infty}\Big(f\Big(\frac{k}{W}\Big)-f\Big(\frac{k}{W}+\delta_k\Big)\Big)\mathrm{si}\{\pi(Wt-k)\}.$$

Whereas the first term is of order $O(\log W/W^{r+\alpha})$ by Theorem 4, that of the second is $O(\delta\log 1/\delta)$ by Theorem 7.

According to Theorems 7 or 8, the sampling expansion also exemplifies stability with respect to the nodal values: a uniformly small error in each of the nodal values produces a correspondingly small error in the recovered signal.

8. Sampling Theorem for Weak Sense Stationary Stochastic Processes

Since signal functions are often of random character, random signals play an important role in signal processing and in the sampling theorem. For this purpose one usually uses stochastic processes which are stationary in the weak sense as a model for them. Given a probability space (Ω, A, P), a stochastic process, namely an A-measurable function $X = X(t) = X(t,\omega)$ of $\omega \in \mathbf{R}$ for each $t \in \mathbf{R}^+$, is said to be weak sense stationary (W. S. S.), if its autocorrelation function (a, c, f,)

$$A_{X,X}(t, t+\tau) := \int_\Omega X(t,\omega) X(t+\tau,\omega)\, dP(\omega)$$

is independent of $t \in \mathbf{R}$, i. e., $A_{X,X}(t,t+\tau) = A_{X,X}(\tau) := A_X(\tau)$. Here it is assumed that X is square integrable with respect to P over Ω, i. e.,

$$E\{|X(t)|^2\} := \int_\Omega |X(t,\omega)|^2 dp(\omega) < \infty \qquad (t \in \mathbf{R}).$$

Such a w. s. s. process X is said to be bandlimited to the interval $[-\pi W, \pi W]$ if the deterministic a. c. f. A_X is bandlimited there. we shall state the sampling theorem for such processes and then examine the time jitter error in more detail.

Theorem 9 *Let X be a w. s. s. process with $E\{|X(t)|^2\} < \infty$, $t \in \mathbf{R}$, such that X is bandlimited to $[-\pi W, \pi W]$. Then*

a)
$$\lim_{N\to\infty} E\{|X(t,\omega) - \sum_{k=-N}^{N} X\left(\frac{k}{W}, \omega\right) \mathrm{si}\{\pi(Wt-k)\}|^2\} = 0.$$

b) If in addition the a. c. f. A_X satisfies (4.2) for some $\gamma \in (0,1]$, then)*

$$[E\{|X(t,\omega) - \sum_{k=-\infty}^{\infty} X\left(\frac{k}{W} + \delta_k, \omega\right) \mathrm{si}\{\pi(Wt-k)\}|^2\}]^{1/2} \tag{8.1}$$

$$\leqslant M_6(A, A_X'', \gamma)\, \delta \log(1/\delta)$$

for $|\delta_k| \leqslant \delta \leqslant \min\{1/W, e^{-2/5}\}$, $k \in \mathbf{Z}$, $W \geqslant 1$, where M_6 is the square root of the constant in (8.6).

Concerning the proof of part a) see [68]. Regarding part b), since X is bandlimited, one can rewrite the square of the left hand side of (8.1) by part a) as

$$\int_\Omega \left| \sum_{k=-\infty}^{\infty} \left(X\left(\frac{k}{W}\right) - X\left(\frac{k}{W} + \delta_k\right)\right) \mathrm{si}\{\pi(Wt-k)\} \right|^2 dP(\omega) \tag{8.2}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \left| \Delta_{-\delta_j}(\Delta_{\delta_k} A_X)\left(\frac{k-j}{W}\right)\right| |\mathrm{si}\{\pi(Wt-k)\} \mathrm{si}\{\pi(Wt-k)\}|$$

*) Here $E\{|f(t) - \sum_{k=-\infty}^{\infty} g_k(t)|^2\}$ is to be understood as $\lim_{N\to\infty} E\{|f(t) - \sum_{k=-N}^{N} g_k(t)|^2\}$

where

$$\Delta_{\pm\delta_k}A_X\left(\frac{l}{W}\right):=A_X\left(\frac{l}{W}\pm\delta_k\right)-A_X\left(\frac{l}{W}\right) \qquad (k,l\in\mathbf{Z}).$$

Under the notation

$$D\left(\frac{l}{W};\delta\right):=\sup_{\substack{|\delta'|<\delta\\|\delta''|<\delta}}\left|\Delta_{\delta'}(\Delta_{\delta''}A_X)\left(\frac{l}{W}\right)\right| \qquad (l\in\mathbf{Z})$$

the integral (8.2) is bounded by

$$(8.3)\quad \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}D\left(\frac{k-j}{W};\delta\right)|\mathrm{si}\{\pi(Wt-j)\}\mathrm{si}\{\pi(Wt-k)\}|$$

$$\leqslant\left\{\sum_{k=-\infty}^{\infty}|\mathrm{si}\{\pi(Wt-k)\}|^q\right\}^{1/q}\left\{\sum_{k=-\infty}^{\infty}\left|\sum_{j=-\infty}^{\infty}D\left(\frac{k-j}{W};\delta\right)|\mathrm{si}\{\pi(Wt-j)\}|\right|^p\right\}^{1/p}$$

in view of Hölder's inequality with $1/p+1/q=1$. Now the latter double sum is bounded on account of the Hausdorff-Young inequality by

$$(8.4)\quad \left\{\sum_{j=-\infty}^{\infty}|\mathrm{si}\{\pi(Wt-j)|^s\right\}^{1/s}\left\{\sum_{j=-\infty}^{\infty}D\left(\frac{j}{W};\delta\right)^r\right\}^{1/r},$$

where $0\leqslant 1/s+1/r-1=1/p$. The right-hand sum in this product is bounded by, noting that $A_X''(t)\in C(\mathbf{R})$,

$$(8.5)\quad (2c(r,\delta)+5)^{1/r}\|A_X''\|_C\delta^2+\left\{\sum_{|j|>C+2}D\left(\frac{j}{W};\delta\right)^r\right\}^{1/r}.$$

where $c=c(r,\delta):=[\delta^{-2/\gamma}W^{r\gamma/(r\gamma-1)}]$. Firstly note that $c(r,\delta)\geqslant 2$ provided $\delta\leqslant e^{-2/5}$, $W\geqslant 1$ and $r\gamma\geqslant 2$. Indeed, $c\geqslant[e^{4/5}W^2]\geqslant[2.225]=2$. Now noting that $W\leqslant(1/\delta)$ for $\gamma\in(0,1]$,

$$\begin{aligned}(2c+5)^{1/r}&<(9c/2)^{1/r}<(9/2)^{1/r}\delta^{-2/r\gamma}W^{\gamma/(r\gamma-1)}\\&\leqslant(9/2)^{1/r}\exp\left\{\left(\frac{2}{\gamma r}+\frac{\gamma}{r\gamma-1}\right)\log(1/\delta)\right\}\\&\leqslant(9/2)^{1/r}\exp\left\{\frac{5}{\gamma r}\log(1/\delta)\right\}\leqslant 2^{\gamma/2}(3/2)^{\gamma}e\end{aligned}$$

if one chooses $r=(5/\gamma)\log(1/\delta)$. The second term in (8.5) is then bounded according to Lemma 2 by

$$2^{1/r}4M_AW^{\gamma}c(r,\delta)^{(1-4\gamma)/r}\leqslant 2^{\gamma/2}4M_A\delta^2e^{2/5}.$$

Setting $s=2r/(2r-1)$, then $p=2r>0$, and the left-hand sum in (8.4) is bounded by $(s')^{1/s}\leqslant s'=2r$, where $1/s+1/s'=1$. Likewise the first sum on the right in (8.3) is bounded by $p=2r$. Combining all the results yields that (8.2) is bounded by

$$(8.6)\quad \frac{2^{\gamma/2}100}{\gamma^2}\{(3/2)^{\gamma}e\|A_X''\|_C+2e^{2/5}M_A\}(\delta\log(1/\delta))^2,$$

uniformly in $t\in\mathbf{R}$. This completes the proof.

Part b) of the foregoing theorem seems to be new. It would be of interest to study the counterpart of part b) for duration limited w. s. s. processes or for not necessarily bandlimited ones. The quantization error for such processes should be a problem of further interest. The methods employed in this paper should be sufficiently powerful to handle the matter. On the other hand, the counterpart of Theorem 4, namely the aliasing error for w. s. s. processes, as well as other interesting generalizations, have already been studied in detail by Splettstösser [65], [68].

General linear stochastic processes which are not necessarily stationary (in the strict or weak sense), nor have independent increments, have been treated in Butzer-Gather [17], however from the point of view of the central limit theorem with rates. These general processes include some random noise as well as pulse train processes as specific models.

9. Further Extensions

It is possible to extend the sampling theorem and the various problems connected with it in many more ways than has been carried out above. Let me indicate just a few of these.

(i) Sampling expansions involving sampled values of the function as well as of its derivative, thus simple Hermite-type interpolation: if $f, f' \in C(\mathbf{R}) \cap L(\mathbf{R})$ and $[f']^\wedge \in L(\mathbf{R})$, then, uniformly in $t \in \mathbf{R}$,

$$f(t) = \lim_{W \to \infty} \sum_{k=-\infty}^{\infty} \left\{ \frac{\sin \frac{\pi}{2}(Wt-k)}{\frac{\pi}{2}(Wt-k)} \right\}^2 \left\{ f\left(\frac{2k}{W}\right) + \left(t - \frac{2k}{W}\right) f'\left(\frac{2k}{W}\right) \right\}.$$

In case of bandlimited functions the limit is dropped. Note that the nodes above are double the distance apart compared with that in (2.1). See [20] for the associated error estimates and the literature.

(ii) The first derivative can be approximated by sampling only the given function: if $f, f' \in C(\mathbf{R}) \cap L(\mathbf{R})$ and $[f']^\wedge \in L(\mathbf{R})$, then, uniformly in $t \in \mathbf{R}$,

$$f'(t) = \lim_{W \to \infty} \pi W \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) \left\{ \frac{\cos \pi(Wt-k)}{\pi(Wt-k)} - \frac{\sin \pi(Wt-k)}{[\pi(Wt-k)]^2} \right\}$$

$$f'(t) = \lim_{W \to \infty} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(-1)^{k+1}}{(k/W)} f\left(t + \frac{k}{W}\right).$$

Also higher order derivatives may be represented in this form; in particular,

$$f''(t) = \lim_{W \to \infty} 2 \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(-1)^{k+1}}{(k/W)^2} f\left(t + \frac{k}{W}\right) + (\pi W)^2 \frac{f(t)}{3}.$$

see [20] and the literature cited there, and Lundin-Stenger [46] for a different approach.

(iii) The Hilbert transform, defined by

$f^{\sim}(t)=\lim_{\delta\to 0}(1/\pi)\int_{|u|\geq\delta}(f(t-u)/u)du$, can be approximated using samples of f: if $f\in C(\mathbf{R})\cap L(\mathbf{R})$, $f^{\wedge}\in L(\mathbf{R})$ and $f^{\sim}\in C(\mathbf{R})$, then (see [20], also [72])

$$f^{\sim}(t)=\lim_{W\to\infty}\sum_{k=-\infty}^{\infty}f\left(\frac{k}{W}\right)\frac{\{\sin(\pi/2)(Wt-k)\}^2}{(\pi/2)(Wt-k)}.$$

For rates in this regard see [70].

(iv) Reconstruction by generalized sampling sums: the function $(\sin t)/t$ in (1.1) is replaced by $g\in C(\mathbf{R})\cap L(\mathbf{R})$ such that $(1/\sqrt{2\pi})\int_R g(u)du=1$, $g^{\wedge}(v)=0$ for $|v|>v$, some $v>0$. Then for each $f\in C_0(\mathbf{R})$ and uniformly in $t\in\mathbf{R}$,

$$f(t)=\lim_{W\to\infty}\frac{\sqrt{(\pi/2)}}{v}\sum_{k=-\infty}^{\infty}f\left(\frac{k}{W}\right)g\left(\frac{\pi}{v}(Wt-k)\right). \tag{9.1}$$

One of the reasons for this procedure is to obtain better rates of approximation than for the classical sampling-series. A concrete example of a function g satisfying the above hypotheses is the kernel function of de La Vallée Poussin in (4.4), namely $g(t):=(3/\sqrt{2\pi})\,\mathrm{si}\,\{3t/2\}\,\mathrm{si}\,\{t/2\}$ for which $v=2$. In this case the rate of approximation in (9.1) is of order $O(1/W^{r+\alpha})$, $r\in\mathbf{P}$, $0<\alpha<1$ if and only if $f^{(r)}\in \mathrm{Lip}(\alpha;C)$. If $\alpha=1$, the saturation case, the Lipschitz class has to be replaced by the Zygmund class. See Stens [74]. A further example is the Fejér kernel $g(t):=\sqrt{1/2\pi}\cdot(\mathrm{si}\{t/2\})^2$ with $v=1$. For each $f\in C_0(\mathbf{R})$, uniformly in $t\in\mathbf{R}$,

$$f(t)=\lim_{W\to\infty}\frac{1}{2}\sum_{W=-\infty}^{\infty}f\left(\frac{k}{W}\right)\left(\mathrm{si}\left\{\frac{\pi}{2}(Wt-k)\right\}\right)^2.$$

Here no condition need be posed upon the Fourier transform nor upon the modulus of continuity. Note that the sum in (9.1) may be regarded as a discretized convolution sum of the associated convolution integral of f and $(\pi W/v)g(\pi Wt/v)$, namely of

$$\frac{1}{\sqrt{2\pi}}\int_R f(u)\frac{\pi W}{v}g\left(\frac{\pi W}{v}(t-u)\right)du \qquad (t\in\mathbf{R}).$$

At this point the extensive work of I. J. Schoenberg and of his many students and collaborators must (again) be emphasized. The difficulty with (1.1) is the slow decay of the function $\mathrm{si}\,\{\pi t\}$ as $t\to\infty$. For this reason one first studied the piecewise linear analogue

$$S^*(t)=\sum_{k=-\infty}^{\infty}f\left(\frac{k}{W}\right)M_2\left(t-\frac{k}{W}\right), \tag{9.2}$$

where $M_2(t)$ is the roof function defined by $M_2(t):=Wt+1$ in $[-1/W,0]$, $M_2(t):=1-Wt$ in $[0,1/W]$, and $M_2(t):=0$ if $|t|>1/W$. Note that $S^*(t)$ is a piecewise linear interpolant with $S^*(k/W)=f(k/W)$, $k\in\mathbf{Z}$. The purpose of the modern field of cardinal spline interpolation now is "to bridge the gap between the piecewise linear $S^*(t)$ defined by (9.2) and the cardinal series (1.1). It aims at retaining some of the sturdiness and simplicity of (9.2), at the same time capturing some of the smoothness and sophistication of (1.1)."For the literature see [55],[56] from which the material may be traced.

So well-known results on singular convolution integrals ([18]) can be applied. For further results in this direction see [62,63], [73,74].

(v) The truncation error, resulting when only a finite number of samples (namely $2N+1$) are used for the representation, has been studied very intensively (see [40]). If $f\in C(\mathbf{R})\cap L(\mathbf{R})$ is bandlimited to $[-\pi W,\pi W]$ such that $[f^\wedge]^{(r)}\in$ Lip$(\alpha;C)$, and $N>\max\{2W|t|,r\}>0$, then

$$\left\|f(t)-\sum_{k=-N}^{N}f\left(\frac{k}{W}\right)\mathrm{si}\{\pi(Wt-k)\}\right\|_C=O\left(\frac{1}{N^{r+\alpha}}\right)\qquad(N\to\infty).$$

This estimate is one of many to be found in U. Scheben [53] and [16]. A typical result in the non-bandlimited case states: If $f\in C(\mathbf{R})\cap L(\mathbf{R})$ satisfies (4.2) for $\nu\in(0,1]$, $f^{(r)}\in\mathrm{Lip}(\alpha;C)$, and $N=[W^{2(1+(r+\alpha)/\nu)}]+1$, then

$$\|f(\cdot)-\sum_{k=-N}^{N}f\left(\frac{k}{W}\right)\mathrm{si}\{\pi(W\cdot-k)\}\|_C=O\left(\frac{\log W}{N^{r+\alpha}}\right)\qquad(W\to\infty).$$

See [21], and for a different approach I. Honda [39].

(vi) Replacement of the trigonometric system by general orthogonal systems on a finite or infinite interval; see H. Kramer [42] and the many papers by A. J. Jerri cited in [40], respectively. In particular, the sampling theorem for the Walsh system, both for sequence-limited and duration-limited functions, is considered e. g. in M. Maqusi [47],[21],[64],Engels-Splettstösser [29]. Results concerning the Haar system which will lead to sampling theory are to be found in [23], [71] and [84]. For the sampling theorem in the Legendre frame on $\mathbf{R}^+$ together with a new type of truncation error estimate see Butzer-Stens-Wehrens [25].

(vii) Sampling theorems for functions with multi-dimensional domain, basic for picture processing and transmission. For such results in the case of the classical trigonometric system see Splettstösser [66] and the extensive literature cited there. For the multi-dimensional Walsh setting see Butzer-Engels [14].

(viii) Implementation of the cardinal series for bandlimited functions: it is possible to replace the transcendental (entire) si-function occurring in the cardinal series by simpler expressions which are easier to compute, namely by certain algebraic polynomials, as a matter of fact by a finite sum of the Taylor expansion of $\mathrm{si}\{\pi(Wt-k)\}$ about the origin. Indeed,

$$(9.3)\qquad \left|\sum_{k=-N}^{N} f\left(\frac{k}{W}\right)\left(\sum_{j=0}^{\rho N-1}(-1)^j\frac{\{\pi(Wt-k)\}^{2j}}{(2j+1)!}\right)-f(t)\right|$$

$$=O\left(\frac{1}{N^{r+\alpha}}\right)+O\left(\frac{1}{\sqrt{N}}\frac{1}{(1.238)^N}\right)$$

for $\rho=5$, $|t|<N/tW$, provided $f\in C(\mathbf{R})\cap L(\mathbf{R})$ is bandlimited to $[-\pi W,\pi W]$ and $[f^\wedge]^{(r)}\in \mathrm{Lip}(\alpha;C), r\in\mathbf{N}$, $0<\alpha\leqslant 1$. So the resulting error consists of the usual truncation error, here of order $O(N^{-r-\alpha})$, plus the additional error of order $O(N^{-1/2}\cdot(1.238)^{-N})$, the error which results from the replacement of the si-function by its Taylor polynomial (of order $5N-1$ with $\rho=5$, which may be regarded as a one-point Hermite interpolation). It is important that the number of terms N of the truncated series be appropriately coupled with $\rho N-1$ terms of the Taylor expansion; in case $\rho=5$ convergence of the (modified) series in (9.3) is guaranteed; if $\rho\leqslant 4$ there are functions for which this series diverges a. e. with order a^N for $a>1$. See Butzer-Engels [15].

(ix) Approximate integration over the real axis: The sampling theorem may be applied to study the approximation of $\int_R f(u)du$ by the Riemann sums $(1/W)\Sigma_{k=-\infty}^{\infty}\cdot f(k/W)$ for $W\to\infty$ together with the associated error estimates. If $f\in C(\mathbf{R})\cap L(\mathbf{R})$ is such that f satisfies (4.2) for some $\gamma>1$, then, for $W>0$,

$$\left|\int_R f(u)du-\frac{1}{W}\sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right)\right|=\sqrt{2\pi}\left|\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} f^\wedge(2k\pi W)\right|$$

whenever the series on the right converges. If $f\in C(\mathbf{R})\cap L(\mathbf{R})$ is such that $\sup\limits_{|h|<\delta}\int_R|f^{(r)}(u+h)-f^{(r)}(u)|du=O(\delta^\alpha)$ for some $r\in\mathbf{N}$ and $0<\alpha\leqslant 1$, then

$$\left|\int_R f(u)du-\frac{1}{W}\sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right)\right|=O\left(\frac{1}{W^{r+\alpha}}\right),$$

see Butzer-Stens [24] and the literature cited there. Such formulae have so far been mainly studied for functions that must instead be holomorphic in some strip around the real axis.

(x) Prediction of bandlimited functions from past samples. Shannon's sampling theorem states that each bandlimited function can be completely reconstructed from its sampled values at a set of equally spaced instants of time on $\mathbf{R}$. For many applications it is important to know whether it is possible to reconstitute such a function from its samples taken exclusively from the past (prediction) via

(9.4) $$f(t)=\lim_{n\to\infty}\sum_{k=1}^{n}a_{kn}f\left(t-\frac{kt}{W}\right)$$

uniformly in $t\in\mathbf{R}$. A closure theorem of N. Levinson (1940) (see also [83]) guarantees the existence of the so-called predictor coefficients a_{kn} (independent of f, t and w) for each $T\in(0,1)$. The basic problem however is the explicit calculation of these a_{kn}. Wainstein and Zubakov [78] calculated them in the form $a_{kn}=(-1)^{k-1}\binom{n}{k}$ provided $T\in(0,1/3)$, and Brown [9] as $a_{kn}=(-1)^{k+1}\binom{n}{k}(\cos\pi T)^k$ provided $T\in(0,1/2)$. Using power series expansions in the complex domain Splettstösser [66, 69] extended T to $T\in(0,1/2)$, respectively $T\in[1/2,2/3)$, with $a_{kn}=(-1)^{k+1}\binom{n+k-1}{k}\cdot\left(\frac{1}{4}\right)^k$, respectively $a_{kn}=-\binom{k-1+n/b}{k}(2\cos\pi T)^k$, where

(9.5) $$b:=\log(1/2+\cos\pi T)/\{\log(-2\cos\pi T)-\log(1/2-\cos\pi T)\}.$$

Splettstösser [66] also generalized the Brown result to functions which are not necessarily bandlimited but whose derivatives of order r satisfy a Lipschitz condition.

The prediction problem can also be treated for random signals; see [66]. Indeed, if $X(t)$ is a w. s. s. process with $E\{|X(t)|^2\}<\infty$, $t\in\mathbf{R}$ that is bandlimited to $[-\pi W,\pi W]$, then

$$\lim_{n\to\infty}E\left\{\left|X(t)-\sum_{k=1}^{n}-\binom{k-1+n/b}{k}(2\cos\pi T)^kX\left(t-\frac{kT}{W}\right)\right|^2\right\}=0$$

uniformly in $t\in\mathbf{R}$, provided $T\in(1/2,2/3)$, where b is defined by (9.5).

Acknowledgement

It is a pleasure to take this opportunity to thank the many members of the "DFG Schwerpunktprogramm Digitale Signalverarbeitung" (Bu 166/29/31/33/36/38), in particular Professors A. Fettweis (Bochum), H. D. Lüke (Aachen), W. Schüssler (Erlangen), H. J. Tafel (Aachen), D. Wolf (Frankfurt) for many stimulating discussions and suggestions during the various meetings of this research group in the course of the past five years which were coordinated in an expert fashion by Dipl-Ing G. Seifert (DFG, Bonn-Bad Godesberg). A number of the problems treated in the survey were posed there. Professor O. Lange (Aachen) kindled the author's first interest in the sampling theorem.

The author would like to extend his special thanks to his research companion Doz. Dr. W. Splettstösser (Aachen) for his unstinting help when writing up the manuscript. The proofs especially of Theorems 5.7 and 9 benefited greatly from his expert advice. Doz. Dr. R. L. Stens (Aachen) read the final version with meticulous care and gave valuable suggestions. He aided especially with Section 5 as well as with the proof of Theorem 2. Finally, the author wishes to thank the DFG for their continued support.

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