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On Einsteinian Manifolds Admitting Orthogonal Families of Totally Umbilical Hypersurfaces*

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Abstract

The aim of the present paper is to study globally the Riemannian manifold admitting two or more mutually orthogonal families of totally umbilical hypersurfaces of which each is Einsteinian. This paper consists of four parts: (i) to establish anew the canonical form of the metric of (M,g) admitting p ($p\geqslant 2$) families of mutually orthogonal totally umbilical hypersurfaces from the standpoint of global differential geometry; (ii) to prove in a n-dimensional ($n>2$) Einsteinian manifold E_n of nonvanishing scalar curvature there doesn't exist one family of compact totally geodesic Einsteinian hypersurfaces (Theorem I); (iii) to prove in a n-dimensional ($n\geqslant 5$) Einsteinian manifold E_n of nonnegative scalar curvature $\bar{R}$ there don't exist two orthogonal families of totally umbilical but not geodesic complete Einsteinian hypersurfaces (Theorem II); (iv) to show that a n-dimensional ($n\geqslant 5$) Riemannian manifold of negative constant scalar curvature $\bar{R}$, admitting p ($p\geqslant 3$) mutually orthogonal families of compact, totally umbilical but not geodesic, Einsteinian hypersurfaces, is of constant curvature, if and only if a number θ (defined in section 5) vanishes (Theorem III).

1 Introduction

Recently Shen Yi-bin proved in [1] a very interesting theorem:

If a Riemannian space V_n of dimension n ($n>3$) admits three orthogonal families of totally umbilical hypersurfaces, among which two families are Einsteinian and the third one are of constant curvature, then the space V_n itself must be of constant curvature.

Shen's proof of this theorem is based upon the following beautiful lemma:

If a Riemannian space (M,g) of dimension n ($n>3$) admits three mutually orthogonal families of totally umbilical Einsteinian hypersurfaces, then the normals to these hypersurfaces are each coincident with a Ricci principal direction of M.

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This lemma will be used in the last section of this paper. I cannot but point out that this lemma is local in nature and answers only partly the question whether a Riemannian space admits three mutually orthogonal families of totally umbilical Einsteinian hypersurfaces. The aim of the present paper is to study this question from the standpoint of global differential geometry. Thereby we have proved some negative results (Theorem I, II, III). We have to emphasize at the beginning that all the manifolds here considered are connected and C^∞ differentiable, and the Riemannian metrics endowed on them are positive definite. We shall concentrate our attention to the Einsteinian manifold, because they are the simplest next to the space of constant curvature.

2 Some fundamental concepts

To deal with globally the problem here considered, some fundamental concepts have to be clarified in advance. Let (M,g) be a Riemannian manifold of dimension n $(n>1)$, and S be a differentiable manifold of dimension n-1. Let I be an open interval $a<t<b$ of $\mathbf{R}$ or an unit circle. Suppose there exists an imbedding Φ of $S\times I$ into M. For any fixed $t_0\in I$, the image of $S\times\{t_0\}$ with the induced metric is a hypersurface of (M,g) and will be denoted hereinafter by S_{t_0}. The collection $\{S_t\,|\,t\in I\}$ is called a family of hypersurfaces admitted by (M,g). According to this definition, all the hypersurfaces belonging to the same family are topologically equivalent.

Now suppose that the Riemannian space (M,g) admits p $(p\geqslant 2)$ families of mutually orthogonal hypersurfaces:

$$\Phi_\alpha:\ S^{(\alpha)}\times I_\alpha\to (M,g)\qquad (\alpha=1,2,\ldots,p)$$

of which each is totally umbilical in M. Let $(x^2,\ldots,x^n)$ be a local coordinate system of $S^{(1)}$ valid in a coordinate neighborhood $U\subset S^{(1)}$. Since Φ_1 is a dif eomorphism of $U\times I_1$ onto $\Phi_1(U\times I_1)$, $\Phi_1(U\times I_1)$ is an open set in M, on which $(x^1,\ldots,x^n)$ $(x^1\in I_1)$ define a coordinate system, and $x^1=$ Const. represents a hypersurface belonging to the first family. The line element of M is then given by $g_{ij}(x)\,dx^i\,dx^j$. Let us integrate the linear differential equation

$$\sum_{j=1}^{n} g^{1j}\frac{\partial f}{\partial x^j}=0$$

and find out $n-1$ independent integrals

$$f_2(x^1,\ldots,x^n),\ldots,f_n(x^1,\ldots,x^n).$$

Since $g^{11}\neq 0$, we have

$$\frac{\partial(f_2,\ldots,f_n)}{\partial(x^2,\ldots,x^n)}\neq 0.$$

Let us write $\bar{x}^1=x^1$, $\bar{x}^j=f_j(x^1,\ldots,x^n)$ for $2\leqslant j\leqslant n$, then $(\bar{x}^1,\ldots,\bar{x}^n)$ form a new local coordinate system of M, in which the line element becomes $\bar{g}_{ij}(\bar{x})d\bar{x}^i\,d\bar{x}^j$ with

$$\bar{g}^{1k} = g^{ij}\frac{\partial \bar{x}^k}{\partial x^i}\frac{\partial \bar{x}^1}{\partial x^j} = g^{1j}\frac{\partial f_k}{\partial x^j} = 0 \qquad (k = 2, \ldots, n),$$

whence $\bar{g}_{1k} = 0$ for $k = 2, \ldots, n$, so that the line element takes the form

$$ds^2 = \bar{g}_{11}(\bar{x})\,(d\bar{x}^1)^2 + \sum_{u,v=2}^{n} \bar{g}_{uv}(\bar{x})\,d\bar{x}^u d\bar{x}^v.$$

Now for simplicity we shall omit without loss of generality the short bars over the letters x and g. If the hypersurfaces $x^1 = \text{Const.}$ are each totally umbilical in M, we can write $g_{uv} = A\ a_{uv}(x^w)$, where $a_{uv}(x^w)$ are functions of $x^2, \cdots, x^n$ only (See [2], p.182.). As to the second family of hypersurfaces

$$\Phi_2 : S^{(2)} \times I_2 \to M,$$

the parameter $t_2 \in I_2$ specifying the hypersurfaces of the second family may be represented as a differentiable function $f(x^1, \ldots, x^n)$. According to our definition, at any point one of the derivatives $\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}$ must be nonvanishing. Now the two hypersurfaces $x^1 = \text{Const.}$ and $f(x^1, \ldots, x^n) = \text{Const.}$ intersect orthogonally, we have

$$\sum_{j=1}^{n} g^{1j}\frac{\partial f}{\partial x^j} = g^{11}\frac{\partial f}{\partial x^1} = 0,$$

and hence f is a function of $x^2, \ldots, x^n$ only. Suppose $\frac{\partial f}{\partial x^2} \neq 0$, we can replace the coordinate x^2 by f, and write it as $\bar{x}^2$ again, then $\bar{x}^2 = \text{Const.}$ are the equations of the second family of hypersurfaces. Meanwhile we should emphasize that according to our construction $\bar{x}^2$ is like x^1 a globally defined coordinate function effective on the whole manifold (M,g). Let us denote by

$$\psi_3(\bar{x}^2, x^3, \ldots, x^n),\ \ldots,\ \psi_n(\bar{x}^2, \ldots, x^n)$$

the n-2 independent integrals of the differential equation

$$a^{22}\frac{\partial \psi}{\partial \bar{x}^2} + a^{23}\frac{\partial \psi}{\partial x^3} + \cdots + a^{2n}\frac{\partial \psi}{\partial x^n} = 0,$$

and use them as the other n-2 new coordinate functions $\bar{x}^3, \ldots, \bar{x}^n$. Then during changing coordinate systems we have

$$\bar{g}^{2t} = \sum_{v=2}^{n} g^{2v}\frac{\partial \psi_t}{\partial x^v} = \frac{1}{A}\sum_{v=2}^{n} a^{2v}\frac{\partial \psi_t}{\partial x^v} = 0$$

for $3 \leqslant t \leqslant n$, whence $\bar{g}_{2t} = 0$. The line element of (M,g) is then reduced to the form

$$ds^2 = g_{11}(x)\,(dx^1)^2 + A\bar{a}_{22}(\bar{x}^2, \cdots, \bar{x}^n)\,(d\bar{x}^2)^2 + A\bar{a}_{rt}(\bar{x}^2, \ldots, \bar{x}^n)\,d\bar{x}^r d\bar{x}^t.$$

where $r, t = 3, \ldots, n$. We omit the short bars over the letters x and a once again. Now the family of hypersurfaces $x^2 = \text{Const.}$ are also totally umbilical, the ratios $\tilde{a}_{rt} : \tilde{a}_{uv}$ $(r,t,u,v = 3, \ldots, n)$ should be functions independent of the variable x^2 ([2], p.182), thus we can write

$$ds^2 = g_{11}(x)(dx^1)^2 + Aa_{22}(x^2,\dots,x^n)(dx^2)^2 + Ba_{rt}(x^3,\dots,x^n)dx^r dx^t$$

with $B \neq 0$. By the same argument we assert

$g_{11} : Ba_{rt}$ = a function independent of x^2;

$Aa_{22} : Ba_{rt}$ = a function independent of x^1.

Accordingly, we write

$g_{11} = B\phi_1(x^1,x^3,\dots,x^n)$, and $Aa_{22} = B\phi_2(x^2,x^3,\dots,x^n)$.

The metric of (M,g) is then reduced to the form

$$ds^2 = B[\phi_1(x^1,x^3,\dots,x^n)(dx^1)^2 + \phi_2(x^2,\dots,x^n)(dx^2)^2 + a_{rt}(x^3,\dots,x^n)dx^r dx^t];$$

If (M,g) admits $p>2$ families of orthogonal totally umbilical hypersurfaces, we can repeat the arguments used above step by step to reduce the metric of (M,g) to the form $(i,j,k = p+1,\dots,n)$

$$ds^2 = e^{2\sigma}\left[\sum_{\alpha=1}^{p} \phi_\alpha(x^\alpha,x^k)(dx^\alpha)^2 + \sum_{i,j=p+1}^{n} a_{ij}(x^k)dx^i dx^j\right],$$

and at the same time $x^1,\dots,x^p$ are each effective on the whole manifold (M,g). Moreover, the equation x^α = Const. represents the totally umbilical hypersurface belonging to the αth family, where $\alpha = 1,\dots,p$.

This result was proved locally by Shen Jin-yuan (former name of Shen Yi-bin) already in 1965 (See [3]). The aim to write here a new proof is to emphasize the global nature and the geometrical meaning of the coordinate functions $x^1,\dots,x^p$. Obviously, the functions σ is defined within an additive function of $x^{p+1},\dots,x^n$. But the partial derivatives $\frac{\partial\sigma}{\partial x^1}, \frac{\partial\sigma}{\partial x^2},\dots,\frac{\partial\sigma}{\partial x^p}$ are uniquely defined on M. These partial derivatives will play an important role in our subsequent arguments, and their geometrical meaning will be explained in section 4.

Now assuming that the hypersurfaces x^α = Const., $1\leqslant\alpha\leqslant p$, be Einsteinian. For the sake of later reference, we summarize some results due to Shen Yi-bin [1] in the form of following.

Lemma 1. *If a Riemannian manifold (M,g) of dimension n $(n>3)$ admits p $(p\geqslant 3)$ mutually orthogonal families of totally umbilical hypersurfaces, of which each is Einsteinian, then the p Ricci curvatures of M along the normals of these hypersurfaces are equal to the same constant λ. Moreover, if we denote by R_α and Ω_α respectively the scalar curvature and the mean curvature of the hypersurface x^α = Const., then the p quantities*

$$\omega_\alpha = R_\alpha - \frac{\Omega_\alpha^2}{n-1}(n-2)$$

are identically equal to the same constant ω, and $\lambda = \frac{1}{2}(\bar{R}-\omega)$.

3 Compact totally geodesic hypersurfaces

Theorem 1. *A n-dimensional $(n>2)$ Einsteinian manifold E_n of nonvanishing scalar* *curvature cannot admit a family of compact totally geodesic Einsteinian hypersurfaces. If the constant scalar curvature of E_n is vanishing, then the family of compact Einsteinian hypersurfaces totally geodesic in E_n must be geodesically parallel, and the space E_n itself must be locally reducible to the product of a hypersurface belonging to the family and a line segment.*

Proof. Let the Einsteinian manifold E_n of nonvanishing constant scalar curvature $\bar{R}$ admit a family of compact $(n-1)$-dimensional submanifolds as its totally geodesic hypersurfaces. Applying the result of the preceding section, the metric of the manifold (M,g) is reducible to the form

$$ds^2 = g_{11}(x)(dx^1)^2 + g_{ij}(x)dx^i dx^j \quad (i,j=2,\dots,n).$$

In view of the formula ([2], p.181, (53.13)) and recalling the coefficients of the second fundamental form $\Omega_{ij}=0$, we find $g_{ij}(x)$ are functions independent of x^1, i.e.,

$$ds^2 = \varphi(x^a)(dx^1)^2 + g_{ij}(x^k)dx^i dx^j, \tag{1}$$

where $a=1,2,\dots,n$; $i,j,k=2,\dots,n$; $\varphi(x^a)>0$; and $g_{ij}(x^k)$ are functions of $x^2,\dots,x^n$ only. As emphasized previously, x^1 is a well defined function specifying the different hypersurfaces of the family, and the orthogonal trajectories to this family are the x^1-curves. Along these curves, $\varphi = \frac{ds}{dx^1}$ and is therefore a well defined smooth function on the whole manifold (M,g). The metric induced on the hypersurface $x^1 =$ Const. is

$$ds_1^2 = g_{ij}(x^k)dx^i dx^j. \tag{2}$$

Since the hypersurface S: $x^1=$Const. is totally geodesic, the coefficients Ω_{ij} of the second fundamental form of S in E_n must be identically zero. By the Gauss-Codazzi equations for a hypersurface, we have

$$R_{hijk} = \bar{R}_{hijk} \quad (h,i,j,k=2,\dots,n),$$

where (R_{hijk}) and $(\bar{R}_{abcd})$ denote the curvature tensors of the hypersurface S and of the ambient space E_n respectively. Moreover, the Ricci tensor of S is given by

$$R_{ij} = -g^{hk}R_{hijk} = -g^{hk}\bar{R}_{hijk} = \bar{R}_{ij} + g^{11}\bar{R}_{1ij1}. \tag{3}$$

and the scalar curvature R of the hypersurface S by

$$R = g^{ij}R_{ij} = g^{ij}\bar{R}_{ij} + g^{ij}g^{11}R_{1ij1} = \bar{R} - 2g^{11}\bar{R}_{11}.$$

Since both E_n and S are Einsteinian, putting

$$R_{ij} = \rho g_{ij},\ \bar{R}_{ab} = \frac{\bar{R}}{n}g_{ab},$$

we have

$$R = (n-1)\rho,\quad R = \frac{n-2}{n}\bar{R},$$

whence

$$\rho=\frac{n-2}{n(n-1)}\bar{R}=\text{Constant}\neq 0.$$

Using (3) and

$$R_{ij}=\frac{R}{n-1}g_{ij},\quad \bar{R}_{ij}=\frac{\bar{R}}{n}g_{ij},$$

we have

$$\bar{R}_{1ij1}=-\frac{\bar{R}}{n(n-1)}g_{11}g_{ij}. \tag{4}$$

But by direct application of a classical formula (See[2], p.20, (8.9)), we find

$$\bar{R}_{1ij1}=\frac{1}{2}\left(\frac{\partial^2\varphi}{\partial x^i\partial x^j}-\left\{{l \atop ij}\right\}\frac{\partial\varphi}{\partial x^l}\right)-\frac{1}{4\varphi}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j}. \tag{5}$$

Equating the right members of (4) and (5) and making the substitution $\psi=\ln\varphi$, we find

$$\psi,_{ij}+\frac{1}{2}\psi,_i\psi,_j=-\frac{2\bar{R}}{n(n-1)}g_{ij}\quad (i,j=2,3,\dots,n), \tag{6}$$

where $\psi,_i$, $\psi,_j$, $\psi,_{ij}$, denote the covariant derivatives of the scalar function ψ with respect to the Riemannian connection induced by the metric (2). Since the hypersurface S is compact, both the supremum and infimum of the function ψ on S are attained at some points A,B respectively. Consequently,

$$\psi,_i(A)=0=\psi,_i(B)\qquad (i=2,3,\dots,n).$$

The matrix $\|\psi,_{ij}\|$ is negative definite at A and positive definite at B. However, the matrix $\|\bar{R}g_{ij}\|$ is always positive definite if $\bar{R}>0$, and always negative definite if $\bar{R}<0$. Thus we end at a contradiction, and the first part of our Theorem 1 is proved.

If $\bar{R}=0$, the equation (6) becomes

$$\psi,_{ij}+\frac{1}{2}\psi,_i\psi,_j=0.$$

Multiplying with g^{ij} and summing in i and j from 2 to n, we find

$$|\text{grad}_s\psi|=g^{ij}\psi,_i\psi,_j=2\Delta\psi.$$

where Δ denotes the Beltrami-Laplacian operator on S. Integrating over the compact hypersurfaces S with volume element dv, we find

$$\int_s|\text{grad}_s\psi|^2dv=2\int_s\Delta\psi dv=0. \tag{7}$$

We conclude from (7)

$$\text{grad}_s\psi=0, \tag{8}$$

and hence $\varphi=\exp\psi$ is a function of x^1 alone. By changing the coordinate function x^1 properly, the metric of E_n is then locally reduced to the form

$$ds^2=(dx^1)^2+g_{ij}(x^k)dx^idx^j. \tag{9}$$

The space E_n is then locally decomposable into the product of a hypersurface and a line segment. The hypersurfaces of the family here considered are obviously geodesically parallel. The proof of our Theorem I is complete.

4 Complete totally umbilical hypersurface

In this section we shall use a theorem due to Morio Obata, let us quote it as (See [4], Theorem A)

Obata Theorem. *In order that a complete Riemannian manifold* (M,g) *of dimension* n $(n>2)$ *admits a nonconstant function* φ *such that* $\nabla_X d\varphi = -c^2\varphi X$ (c *is a positive constant*) *holds true for any tangent vector* X, *the necessary and sufficient condition is that this manifold should be isometric globally with a* n*-sphere of radius* $1/c$ *in an euclidean space of dimension* $n+1$.

Even though this is a global theorem, yet in any coordinate neighborhood of the coordinate functions $(x^1,\dots,x^n)$, the condition

$$\nabla_X d\varphi = -c^2\varphi X \tag{10}$$

is equivalent to

$$\varphi_{,ij} = -c^2\varphi g_{ij} \quad (i,\ j=1,\dots,n)$$

where (g_{ij}) is the metric tensor in the specified coordinate system, and $\varphi_{,ij}$ is the second covariant derivative of the scalar function φ with respect to the Riemannian connection derived from (g_{ij}). In fact, (10) is equivalent to

$$\nabla_{\frac{\partial}{\partial x^i}}\left(g^{kj}\frac{\partial\varphi}{\partial x^j}\ \frac{\partial}{\partial x^k}\right) = -c^2\varphi\frac{\partial}{\partial x^i} \tag{11}$$

for $i=1,2,\dots,n$. Recalling

$$\frac{\partial}{\partial x^i}\ g^{kj} = -g^{kj}\begin{Bmatrix} k \\ ih\end{Bmatrix} - g^{kh}\begin{Bmatrix} j \\ ih\end{Bmatrix},$$

$$\nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^k}\right) = \begin{Bmatrix} l \\ ik\end{Bmatrix}\frac{\partial}{\partial x^l},$$

and

$$\varphi_{,ij} = \frac{\partial^2\varphi}{\partial x^i\partial x^j} - \begin{Bmatrix} k \\ ij\end{Bmatrix}\frac{\partial\varphi}{\partial x^k},$$

and developing the left side of (11), we get

$$\varphi_{,ij} = -c^2\varphi g_{ij} \quad (i,j=1,2,\dots,n) \tag{12}$$

immediately. Since all the calculations are reversible, (11) is also a consequence of (12). Thus we can say (12) is a local representation of (11).

Now we are in position to prove our

Theorem II. *If the constant scalar curvature* $\bar{R}$ *of an Einsteinian manifold* E_n *of dimension* n $(n\geqslant 5)$ *is non-negative, then* E_n *cannot admit* p $(p\geqslant 2)$ *families of orthogonal, totally umbilical but not geodesic, complete hypersurfaces of which each is Einsteinian.*

Proof. Evidently, if merely for the purpose of proving our Theorem Ⅱ, we need to consider the case $p=2$ only. But the general formulas which we shall derive are useful in the sequel, let us still assume $p\geqslant 2$, For convenience, we shall use the following notations:

Ω_τ = Mean curvature of the hypersurface x^α = Const., $\bar{R}$ = Scalar curvature of the space E_n,

$$\sigma_\alpha=\frac{\partial\sigma}{\partial x^\alpha},\quad \sigma_{\alpha\beta}=\frac{\partial^2\sigma}{\partial x^\alpha\partial x^\beta},$$

and

$$H=\frac{1}{(n-1)^2}\left\{\Omega_1^2+\dots+\Omega_p^2+\frac{n-1}{n}\bar{R}\right\}.$$

Since E_n is assumed being an Einsteinian manifold, and admitting p ($p\geqslant 2$) families of orthogonal totally umbilical hypersurfaces, the normals to these hypersurfaces trivially coincide with the Ricci principal directions of E_n, and hence the mean curvature Ω_α depends upon the variable x^α only ([5], p.99). According to our result in section 2, the line element of E_n is reducible to the form

$$ds^2=e^{2\sigma}\left\{\sum_{\alpha=1}^{p}\varphi_\alpha(x^\alpha,x^k)(dx^\alpha)^2+a_{ij}(x^k)dx^idx^j\right\},\tag{13}$$

where $i,j,k=p+1,\dots,n$ and the coordinate functions $x^1,\dots,x^p$ as well as the functions $\frac{\partial\sigma}{\partial x^\alpha}$ are defined globally on the manifold E_n. The unit normal vector to the hypersurface x^α = Const. at a generic point may be represented by the components

$$\xi^a_{\alpha 1}=\frac{1}{e^\sigma\sqrt{\varphi_\alpha}}\delta^a_\alpha\quad(a=1,2,\dots,n),$$

and the coefficients of the second fundamental form (See [2], p.148, (43.7)) are

$$\Omega^{(\alpha)}_{ij}=-\frac{1}{e^\sigma\sqrt{\varphi_\alpha}}a_{ij}e^{2\sigma}\sigma_\alpha\quad(i,j,=p+1,\dots,n).$$

On the other hand, since every point on the hypersurface x^α = Const. is an umbilical point, the relation

$$\Omega^{(\alpha)}_{ij}=\frac{\Omega_\alpha}{n-1}e^{2\sigma}a_{ij}$$

holds true. Hence we have

$$\varphi_\alpha=\left(\frac{n-1}{\Omega_\alpha}\right)^2e^{-2\sigma}\sigma_\alpha^2,$$

and (13) can be reduced to the form

$$ds^2=\sum_{\alpha=1}^{p}\left(\frac{n-1}{\Omega_\alpha}\right)^2\sigma_\alpha^2(dx^\alpha)^2+e^{2\sigma}a_{ij}(x^k)dx^idx^j.\tag{14}$$

Since M is connected and $\varphi_\alpha\neq 0$, without loss of generality we may assume $\sigma_\alpha>0$.

On the x^α-curves σ_α and the differential of arc length ds are connected by the relation

$$\sigma_\alpha = \frac{\Omega_\alpha}{n-1}\frac{ds}{dx^\alpha},$$

which provides a geometrical interpretation of the function σ_α. Besides, when $1 \leqslant \beta \leqslant p$, and $\beta \neq \alpha$, the function

$$e^{-\sigma}\sigma_\alpha = \frac{\Omega_\alpha}{n-1}\sqrt{\varphi_\alpha}$$

is independent of x^β. Hence we have a relation constantly used in the sequel:

$$\sigma_{\alpha\beta} = \frac{\partial^2 \sigma}{\partial x^\alpha \partial x^\beta} = \sigma_\alpha \sigma_\beta \qquad (\alpha \neq \beta)$$

Let us write for brevity

$$c_\alpha = \left(\frac{n-1}{\Omega_\alpha}\right)^2, \text{ a function of } x^\alpha \text{ only,}$$

and

$$g_{\alpha\alpha} = c_\alpha \sigma_\alpha^2, \quad g_{ij} = e^{2\sigma} a_{ij}(x^k),$$

where $\alpha = 1, 2, \ldots, p$; $i, j = p+1, \ldots, n$, and all the components $g_{\alpha i} = 0$. Now using the Lemma I of section 2, we have

$$R_\alpha - \frac{\Omega_\alpha^2}{n-1}(n-2) = \omega, \quad \bar{R}_{ij} = \frac{\bar{R}}{n} g_{ij}, \quad \lambda = \frac{\bar{R}}{n},$$

for $i, j = p+1, \ldots, n$. Then by the same arguments as we have used to derive (4), we find easily

$$\bar{R}_{\alpha i j \alpha} = -\frac{\bar{R}}{n(n-1)} g_{ij} g_{\alpha\alpha}. \tag{15}$$

When $p+1 \leqslant i, j \leqslant n$, through direct application of the formula (8, 9) of [2, p. 20] and making use of the relation $\sigma_{\alpha\beta} = \sigma_\alpha \sigma_\beta$ for $\alpha \neq \beta$ to simplify the result, we find

$$\bar{R}_{\alpha i j \alpha} = c_\alpha \sigma_\alpha \left\{ \sigma_{\alpha, ij} + \left(\sigma_\alpha \sum_{\beta=1}^{p} \frac{1}{c_\beta} - \frac{1}{2}\frac{c'_\alpha}{c_\alpha^2} \right) g_{ij} \right\}, \tag{16}$$

where

$$\sigma_{\alpha, ij} = \frac{\partial^2 \sigma_\alpha}{\partial x^i \partial x^j} - \frac{\partial \sigma_\alpha}{\partial x^l}\left\{ {l \atop ij} \right\}$$

with $i, j, l = p+1, \ldots, \ldots, n$. Equating the right members of (15) and (16), we find

$$\sigma_{\alpha, ij} = -H\left(\sigma_\alpha + \frac{\Omega_\alpha \Omega'_\alpha}{(n-1)^2 H}\right) g_{ij}. \tag{17}$$

Define

$$\psi_\alpha = \sigma_\alpha + \frac{\Omega_\alpha \Omega'_\alpha}{H(n-1)^2}.$$

Then making use of the fact that $\Omega_1, \ldots, \Omega_p$ and H are independent of the variables

$x^{p+1},\ldots,x^n$, we have a trivial relation $\psi_{a,ij}=\sigma_{a,ij}$ for i, $j=p+1,\ldots,n$. In this way (17) is reduced to

$$\psi_{a,ij}=-H\psi_a g_{ij}. \tag{18}$$

According to our assumption that all the p hypersurfaces, one taken from each family:

$$x^1=\text{Const.},\ x^2=\text{Const.},\ldots,\ x^p=\text{Const.},$$

are complete, their intersections must be complete also. To any one of these intersections, say Q, of dimension n-p, we can apply Obata's theorem. If for certain $a(1\leqslant a\leqslant p)$, the functions ψ_a were not identically zero, taking account of (18), ψ_a must be a nonconstant function. However, H is a positive constant on the intersection Q, because it depends upon the variables $x^1,\ldots,x^p$ only. Therefore, according to Obata's theorem, Q must be isometric globally with a $(n-p)$-sphere of radius $1/\sqrt{H}$. Let us take $a_{ij}(x^k)dx^i dx^j$ to be the canonical metric of a unit (n-p)-sphere, then

$$e^{2\sigma}a_{ij}(x^k)dx^i dx^j,$$

being the fundamental form of a $(n-p)$-sphere of radius $1/\sqrt{H}$, must be identical with $\frac{1}{H}a_{ij}(x^k)dx^i dx^j$, and hence $\exp(2\sigma)=1/H$. Thus

$$\sigma_a=-\frac{1}{2}\frac{\partial}{\partial x^a}\ln H=-\frac{\Omega_a\Omega'_a}{\sum_{a=1}^{p}\Omega_a^2+\frac{n-1}{n}\bar{R}}=-\frac{\Omega_a\Omega'_a}{(n-1)^2H},$$

and $\quad \psi_a=\sigma_a+\dfrac{\Omega_a\Omega'_a}{H(n-1)^2}=0.$

This result is contradictory to our assumption $\psi_a\neq0$. Thus we are forced to conclude

$$\psi_1=0=\psi_2=\cdots=\psi_p,$$

and hence

$$\sigma_a=-\frac{\Omega_a\Omega'_a}{(n-1)^2H}\qquad \text{for } a=1,\cdots,p.$$

By the relations $\sigma_{a\beta}=\sigma_a\sigma_\beta$, we find at least one of the functions $\Omega'_1,\Omega'_2,\cdots,\Omega'_p$ must be zero, and consequently, one of $\sigma_1,\sigma_2,\cdots,\sigma_p$ must be zero. But this is impossible, since the metric of E_n is assumed positive definite. By these contradictions our Theorem II is completely established.

5 Riemannian manifold of negative constant scalar curvature

It may be interesting to investigate when a Riemannian manifold admits p $(p>2)$ mutually orthogonal families of compact totally umbilical but not geodesic Einsteinian hypersurfaces. For this purpose let us suppose that the manifold (M,g) be of negative constant scalar curvature $\bar{R}$, this case is important in view of our Theorem II. We shall prove the following.

Theorem III. *Let a n-dimensional* $(n>3)$ *complete and simply connected manifold* (M,g) *of negative constant scalar curvature* $\bar{R}$ *admit* p $(p\geq 3)$ *orthogonal families of compact, totally umbilical but not geodesic hypersurfaces, of which each is Einsteinian. Then* (M,g) *is a space of constant curvature, if and only if a number* $\theta=0$, *where*

$$\theta=\bar{R}-\frac{n-p}{n-1}\omega+\sum_{\beta=1}^{p}\frac{n-p}{(n-1)^2}\Omega_\beta^2-p\lambda,$$

and $\omega,\lambda,\Omega_\beta$ *are interpretated as in Lemma I.*

Proof. Necessity of the condition $\theta=0$ is almost trivial. For (M,g) has been assumed complete and simply connected. If it is of constant curvature $K<0$, according to Cartan-Hadamard theorem it is diffeomorphic to $\mathbf{R}^n$, and we can write its metric in the form

$$ds^2=\frac{\sum_{i=1}^{n}(dx^i)^2}{\left[1+\frac{K}{4}\sum_{i=1}^{n}(x^i)^2\right]^2}$$

with $\sum_{i=1}^{n}(x^i)^2<-4/K$. In such a manifold each one of the hypersurfaces belonging to the family

$$x^i=\text{Const. c}\qquad (|c|>2/\sqrt{-K})$$

is totally umbilical. It is easy to verify that these hypersurfaces have constant curvature $K(1+\frac{1}{4}K\ c^2)>0$. Since each one of them is complete, they must be compact ([6], p.28). The index i may be any integer from 1 to n, and therefore the space M of negative constant curvature admits n families of compact mutually orthogonal hypersurfaces, of which each is Einsteinian, totally umbilical but not geodesic in M. That means in our case $p=n$, and $\theta=\bar{R}-n\lambda=0$, because λ is the Ricci curvature of (M,g).

Now we are going to show that the condition $\theta=0$ is also sufficient. For the case we are considering, we apply the Gauss equation

$$R_{hijk}=(\Omega_{ik}\Omega_{hj}-\Omega_{hk}\Omega_{ij})+\bar{R}_{hijk}\qquad (h,i,j,k=p+1,\dots,n)$$

to the totally umbilical Einsteinian hypersurfaces $x^\alpha=\text{Constant}$ $(1\leq\alpha\leq p)$ and then contract the indices h and k, we find

$$\frac{R_\alpha}{n-1}g_{ij}=\left(\frac{\Omega_\alpha}{n-1}\right)^2(n-2)g_{ij}+\bar{R}_{ij}+\frac{1}{g_{\alpha\alpha}}\bar{R}_{\alpha ij\alpha}.\tag{19}$$

Combining (16) and (19) and using the relations

$$c_\alpha=\left(\frac{n-1}{\Omega_\alpha}\right)^2,\quad g^{ij}\bar{R}_{ij}=\bar{R}-p\lambda,\quad g_{\alpha\alpha}=c_\alpha\sigma_\alpha^2,$$

we find

$$\sigma_{\alpha,ij}=\left(\frac{\omega}{n-1}g_{ij}-\bar{R}_{ij}\right)\sigma_\alpha-\frac{\sigma_\alpha}{(n-1)^2}\sum_{\beta=1}^{p}\Omega_\beta^2\ g_{ij}-\frac{\Omega_\alpha\Omega'_\alpha}{(n-1)^2}g_{ij}.$$

Multiplying this equation with g^{ij} and summing in i and j from $p+1$ to n, we obtain

$$g^{ij}\sigma_{\alpha,ij} = -\theta\sigma_\alpha - \frac{n-p}{(n-1)^2}\Omega_\alpha\Omega'_\alpha, \tag{20}$$

where

$$\theta = \bar{R} - \frac{n-p}{n-1}\omega + \sum_{\beta=1}^{p}\frac{n-p}{(n-1)^2}\Omega_\beta^2 - p\lambda \tag{21}$$

is a function of $x^1,\dots,x^p$ only. By means of the relations (See the Lemma I)

$$\Omega_\beta^2 = \frac{n-1}{n-2}(R_\beta - \omega), \quad \lambda = \frac{1}{2}(\bar{R} - \omega),$$

(21) is reduced to

$$\theta = -\frac{1}{2}(p-2)\bar{R} + \frac{n-p}{(n-1)(n-2)}\sum_{\beta=1}^{p}R_\beta + \left(\frac{p}{2} - \frac{n-p}{n-1} - \frac{p(n-p)}{(n-1)(n-2)}\right)\omega. \tag{22}$$

Note that R_β is a function depending upon x^β only.

Define

$$\tilde{\Delta}\sigma_\alpha = -g^{ij}\sigma_{\alpha,ij},$$

where the indices i, j run from $p+1$ to n, and rewrite (20) in the form

$$\tilde{\Delta}\sigma_\alpha = \theta\sigma_\alpha + \frac{n-p}{(n-1)^2}\Omega_\alpha\Omega'_\alpha. \tag{23}$$

Now imposing the assumption $\theta = 0$ on the relation (23) and then integrating its reduced form

$$\tilde{\Delta}\sigma_\alpha = \frac{n-p}{(n-1)^2}\Omega_\alpha\Omega'_\alpha \tag{24}$$

over the intersection Q of the p hypersurfaces:

$$x^1 = \text{Const.}, \dots, x^p = \text{Const.},$$

we find $\Omega'_\alpha = 0$. and Ω_α = constant for $\alpha = 1, \dots, p$. By the way we have to remind that the n-p dimensional submanifold Q is compact, because the hypersurfaces x^α = Const. have been assumed compact. Then (24) is reduced to $\tilde{\Delta}\sigma_\alpha = 0$ and hence we obtain

$$0 = \int_Q \sigma_\alpha\tilde{\Delta}\sigma_\alpha dv = \int_Q g^{ij}\sigma_{\alpha,j}\sigma_{\alpha,i}dv.$$

Consequently $\sigma_{\alpha,j} = 0$, and σ_α is a function depending upon $x^1,\dots,x^p$ only. As a result, we can write

$$e^\sigma = \exp f(x^1,\dots,x^p)\,\Theta(x^{p+1},\dots,x^n).$$

On the other hand, integrating the equation $\sigma_{\alpha\beta} = \sigma_\alpha\sigma_\beta$ with respect to x^β $(\beta \neq \alpha)$, we deduce

$$\sigma_\alpha = e^\sigma\phi_\alpha(x^\alpha, x^k) = e^f\psi_\alpha(x^\alpha, x^k). \tag{25}$$

In view of $\sigma_{\alpha,j} = 0$, ψ_α must be a function of x^α alone. Then making use of $\sigma_{\alpha\beta} = \sigma_\alpha\sigma_\beta$ again, from $\sigma_\alpha = e^f\psi(x^\alpha)$ we find

$$\frac{\partial f}{\partial x^\beta} = e^f\psi_\beta(x^\beta) \quad (\beta = 1,2,\dots,p) \tag{26}$$

and

$$g_{\alpha\alpha}=c_\alpha\sigma_\alpha^2=\left(\frac{n-1}{\Omega_\alpha}\right)^2 e^{2f}\left[\psi_\alpha(x^\alpha)\right]^2.$$

By changing the coordinate functions $x^1,\dots,x^p$ properly, the metric of (M,g) is reducible to the form

$$ds^2=e^{2f}\left[\sum_{\alpha=1}^{p}(dx^\alpha)^2+a_{ij}(x^k)dx^idx^j\right], \tag{27}$$

in which f is a function of $x^1,\dots,x^p$ only, and $i,j,k=p+1,\dots,n$. To penetrate a step further, let us calculate the Ricci tensor of the hypersurface

$$x^1=\text{Const. } c$$

of which the induced metric is

$$d\tilde{s}^2=e^{2\tilde{f}}\left[\sum_{\alpha=2}^{p}(dx^\alpha)^2+a_{ij}(x^k)dx^idx^j\right] \tag{27*}$$

with $\tilde{f}=f|_{x^1=c}$. Using the formula (28.6) on the page 90 of [2] to calculate the components of the Ricci tensor of the hypersurface $x^1=c$, we find

$$-\tilde{R}^{(1)}_{\alpha\beta}=(n-3)(\tilde{f}_{,\alpha\beta}-\tilde{f}_{,\alpha}\tilde{f}_{,\beta})+\delta_{\alpha\beta}(\Delta_2\tilde{f}+(n-3)\Delta_1\tilde{f}) \tag{28}$$

and

$$-\tilde{R}^{(1)}_{\alpha i}=0\quad(\alpha,\beta=2,3,\dots,p;\ i=p+1,\dots,n),$$

where

$$\tilde{f}_{,\alpha}=\frac{\partial\tilde{f}}{\partial x^\alpha},\quad \tilde{f}_{,\alpha\beta}=\frac{\partial^2\tilde{f}}{\partial x^\alpha\partial x^\beta},\quad \Delta_1\tilde{f}=\sum_{\alpha=2}^{p}\left(\frac{\partial\tilde{f}}{\partial x^\alpha}\right)^2,\quad \Delta_2\tilde{f}=\sum_{\alpha=2}^{p}\frac{\partial^2\tilde{f}}{(\partial x^\alpha)^2}.$$

By means of (26), we find $\tilde{R}^{(1)}_{\alpha\beta}=0$ for $\alpha\neq\beta$. In order to make $\tilde{R}^{(1)}_{\alpha\alpha}=\tilde{R}^{(1)}_{\beta\beta}$ for $\alpha\neq\beta$, we have necessarily

$$\frac{\partial^2\tilde{f}}{\partial x^\alpha\partial x^\alpha}-\left(\frac{\partial\tilde{f}}{\partial x^\alpha}\right)^2=\frac{\partial^2\tilde{f}}{\partial x^\beta\partial x^\beta}-\left(\frac{\partial\tilde{f}}{\partial x^\beta}\right)^2. \tag{29}$$

Combining (26) and (29), we find

$$\frac{\partial\psi_\alpha}{\partial x^\alpha}=\frac{\partial\psi_\beta}{\partial x^\beta}\qquad\text{for } \alpha\neq\beta.$$

Since ψ_α depends upon x^α alone, from this equation we deduce

$$\frac{\partial\psi_\alpha}{\partial x^\alpha}=\text{a constant } 2a$$

for $\alpha=2,3,\dots,p$. Thus $\psi_\alpha=2a_1x^\alpha$. Then by integrating (26) we obtain

$$-e^{-f}=a_1\sum_{\alpha=1}^{p}(x^\alpha)^2+\phi_1(x^1), \tag{30}$$

where $\phi_1(x^1)$ is a function of x^1 to be determined. The same calculation can be applied to the other p-1 families of hypersurfaces $x^\beta=\text{Const.}$ $(\beta=2,\dots,p)$, and it follows

$$-e^{-f}=a_\beta\sum_{\alpha=1}^{p}(x^\alpha)^2+\phi_\beta(x^\beta). \tag{31}$$

By comparing the coefficients of x^γ $(\gamma\neq 1,\beta)$ in (30) and (31), we find $a_1=a_\beta$, and hence $\phi_1(x^1)=\phi_\beta(x^\beta)=\text{Const.}$ Thus $a_1=\cdots=a_p=\text{Const. } a$; $\phi_1(x^1)=\cdots=\phi_p(x^p)=\text{Const.}$ b. Now we can write

$$-e^{-f}=a\sum_{\alpha=1}^{p}(x^\alpha)^2+b,$$

and for the hypersurface $x^1=c$ we have

$$-e^{-\tilde{f}}=a\sum_{\alpha=2}^{p}(x^\alpha)^2+ac^2+b.$$

Substituting this result into the expression for $\tilde{R}^{(1)}_{\alpha\alpha}(\alpha=1,2,\ldots,p)$, the components $\tilde{R}^{(1)}_{\alpha\alpha}$ of the Ricci tensor for the hypersurface $x^1=$ Const. c are found to be

$$-\tilde{R}^{(1)}_{\alpha\alpha}=e^{2\tilde{f}}\Big[2a^2(n-p)\sum_{\alpha=2}^{p}(x^\alpha)^2-2a(n+p-4)(ac^2+b)\Big].$$

For the hypersurface $x^1=$ Const. to be Einsteinian, it is necessary that (recalling $n-1>2$) for $\alpha=2,\ldots,p$

$$\tilde{R}^{(1)}_{\alpha\alpha}=e^{2\tilde{f}}\times(\text{constant})\qquad(\text{use }(27^*)).$$

therefore we have either $p=n$ or $a=0$.

(i) $p=n$. In this case, the metric of (M, g) is reducible to

$$ds^2=\frac{\sum_{i=1}^{n}(dx^i)^2}{\Big[a\sum_{\alpha=1}^{n}(x^\alpha)^2+b\Big]^2},$$

of which the sectional curvature is a negative constant $K_0=4ab$.

(ii) $a=0$. The metric of (M,g) is then reduced to

$$ds^2=\frac{1}{b^2}\Big[(dx^1)^2+\cdots+(dx^n)^2\Big],$$

which is flat. Hence the hypersurface $x^\alpha=$ Const. $(1\leqslant\alpha\leqslant p)$ would be totally geodesic. However, this situation has been excluded from our discussion. The proof of our Theorem III is complete.

Question. In the case $\theta\neq0$, does there exist a Riemannian space which admits p ($p\geqslant3$) orthogonal families of compact, totally umbilical but not geodesic, Einsteinian hypersurfaces?

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