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On Modular Hilbert Algebras*

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In this paper, we shall make some discussions on Tomita-Takesaki's fundamental theorem, and point out that a modular Hilbert algebra which fails to satisfy the condition (Ⅷ) has an extension of modular Hilbert algebra, and then we shall answer the question: "Is a #-subalgebra of a modular Hilbert algebra still a modular Hilbert algebra?" The partially affirmative result is that: "a #-two-sided ideal of a modular Hilbert algebra is still a modular Hilbert algebra."

§1 Tomita—Takesaki's fundamental theorem

Theorem([1]) For every generalized Hilbert algebra $\mathscr{A}$, there exists a modular Hilbert algebra $\mathfrak{B}$ which is equivalent to $\mathscr{A}$.

In this section, we shall make some discussions on such $\mathfrak{B}$.

let $(\mathscr{A}, \#, \langle,\rangle)$ be a complete generalized Hilbert algebra, $\mathscr{H}$ be the completion of $(\mathscr{A}, \langle,\rangle)$, Δ be its modular operator. A #-subalgebra $\mathfrak{B}$ of $\mathscr{A}$ is called a modular Hilbert algebra equivalent to $\mathscr{A}$, if $\mathfrak{B}''=\mathscr{A}$, $\Delta^{\alpha}\mathscr{B}\subset\mathscr{B}\,(\forall\alpha\in C)$. and $(\mathscr{B}, \#, \Delta(\alpha)=\Delta^{\alpha}, \langle,\rangle)$ is a modular Hilbert algebra.

Lemma 1.1 let $\mathscr{D}=\bigcap\{\mathscr{D}(\Delta^{\alpha})\,|\,\alpha\in\mathbb{C}\}$, then Δ^{α} is the closure of $\Delta^{\alpha}|\mathscr{A}'\cap\mathscr{D}$, $\forall\alpha\in\mathbb{C}$.

Proof From[1], for every $\alpha\in\mathbb{C}$, Δ^{α} is the closure of

$$\Delta^{\alpha}|\{f(\log\Delta)\xi\,|\,\xi\in\mathscr{A},\ f\in\mathscr{E}\}.$$

But $f(\log\Delta)\xi\in\mathscr{A}\cap\mathscr{D}$, $\forall\xi\in\mathscr{A}$, $f\in\mathscr{E}$, hence Δ^{α} is the closure of $\Delta^{\alpha}|\mathscr{A}\cap\mathscr{D}$.

Because Δ^{-1} is the modular operator of $\mathscr{A}'$, so we have symmetrically that Δ^{α} is the closure of $\Delta^{\alpha}|\mathscr{A}'\cap\mathscr{D}$, $\forall\alpha\in\mathbb{C}$. Q. E. D.

Proposition 1.2 Let

$$\mathscr{U}=\{\xi\,|\,\xi\in\mathscr{A}\cap\mathscr{D},\ \Delta^{\alpha}\xi\in\mathscr{A},\ \forall\alpha\in\mathbb{C}\},$$

then $\mathscr{U}$ is the maximum modular Hilbert algebra equivalent to $\mathscr{A}$.(*)

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(*) Another Proof see[2].

Proof If $\xi,\eta\in\mathcal{U}$, $\zeta\in\mathcal{S}'\cap\mathcal{D}$, then

$$\langle\xi\eta,\Delta^{-it}\zeta\rangle=\langle\Delta^{it}(\xi\eta),\zeta\rangle=\langle\Delta^{it}\Pi(\xi)\Delta^{-it}\Delta^{it}\eta,\zeta\rangle$$
$$=\langle\Delta^{it}\eta,\Pi((\Delta^{it}\xi)^*)\zeta\rangle=\langle\Delta^{it}\eta,\Pi'(\zeta)\Delta^{it}\xi^*\rangle,\ \forall t\in\mathbb{R}$$

On other hand, $\alpha\to\langle\xi\eta,\Delta^{\bar\alpha}\zeta\rangle$ and $\alpha\to\langle\Delta^{\alpha}\eta,\Pi'(\zeta)\Delta^{-\bar\alpha}\xi^*\rangle$ are analytic on $\mathbb{C}$, so for every $\alpha\in\mathbb{C}$

$$\langle\xi\eta,\Delta^{\bar\alpha}\zeta\rangle=\langle\Delta^{\alpha}\eta,\Pi'(\zeta)\Delta^{-\bar\alpha}\xi^*\rangle=\langle\Delta^{\alpha}\eta,\Pi((\Delta^{\alpha}\xi)^*)\zeta\rangle=\langle(\Delta^{\alpha}\xi)(\Delta^{\alpha}\eta),\zeta\rangle$$

Now by lemma 1.1, we have

$$\Delta^{\alpha}(\xi\eta)=(\Delta^{\alpha}\xi)(\Delta^{\alpha}\eta),\qquad\forall\xi,\eta\in\mathcal{U},\alpha\in\mathbb{C}$$

However, from[1], $(1+\Delta^t)\{f(\log\Delta)\xi|\xi\in\mathcal{S},\ f\in\mathcal{E}\}$ is dense in $\mathcal{H}$ $(\forall t\in\mathbb{R})$, hence $(1+\Delta^t)\mathcal{U}$ is dense in $\mathcal{H}$ $(\forall t\in\mathbb{R})$. Therefore $\mathcal{U}$ is the maximum modular Hilbert algebra equivalent to $\mathcal{S}$. Q. E. D.

Proposition 1.3 let $\mathcal{U}_a$ be the #-subalgebra generated by

$$\left\{\xi_r=\sqrt{\frac{r}{\pi}}\int_{-\infty}^{+\infty}e^{-rt^2}\Delta^{it}\xi dt\,|\,\xi\in\mathcal{S},r>0\right\}$$

Then $\mathcal{U}_a$ is a modular Hilbert algebra equivalent to $\mathcal{S}$.

Proof From[3], ξ_r is an analytic vector respect to $\{\Delta^{it}\}$, so $\xi_r\in\mathcal{D}$, $\forall\xi\in\mathcal{S}$, $r>0$. On the other hand

$$\Pi'(\eta)\xi_r(\alpha)=\sqrt{\frac{r}{\pi}}\int_{-\infty}^{+\infty}e^{-r(t-\alpha)^2}\Pi'(\eta)\Delta^{it}\xi dt=\left(\sqrt{\frac{r}{\pi}}\int_{-\infty}^{+\infty}e^{-r(t-\alpha)^2}\Delta^{it}\Pi(\xi)\Delta^{-it}dt\right)\eta,\quad\forall\eta\in\mathcal{S}'$$

where

$$\xi_r(\alpha)=\sqrt{\frac{r}{\pi}}\int_{-\infty}^{+\infty}e^{-r(t-\alpha)^2}\Delta^{it}\xi dt=\Delta^{i\alpha}\xi,\quad\forall\alpha\in\mathbb{C}$$

so $\xi_r(\alpha)$ is a left bounded element. Furthermore $\xi_r(\alpha)\in\mathcal{D}(\Delta^{\frac{1}{2}})$, so $\xi_r(\alpha)\in\mathcal{S}$. Therefore $\mathcal{U}_a\subset\mathcal{U}$.

Now it is sufficient to prove that $(1+\Delta^t)\mathcal{U}_a$ is dense in $\mathcal{H}$, $\forall t\in\mathbb{R}$. For every $\xi\in\mathcal{D}(\Delta^t)$ and $\delta>0$, by[1], there are $f\in\mathcal{E}$ and $\eta\in\mathcal{S}$ such that

$$\|(1+\Delta^t)(f(\log\Delta)\eta-\xi)\|<\delta$$

The operator $(1+\Delta^t)f(\log\Delta)$ is bounded and $\|\eta_r-\eta\|\to0\ (r\to+\infty)$, so

$$\|(1+\Delta^t)(f(\log\Delta)\eta_r-\xi)\|=\|(1+\Delta^t)[(f(\log\Delta)\eta)_r-\xi]\|<\delta$$

when r is sufficiently large. This completes the proof.

Proposition 1.4 Let

$$\mathcal{U}_0=\left\{\xi\,\middle|\,\begin{array}{l}\xi\in\mathcal{U},\text{ and }\alpha\to\Pi(\Delta^{\alpha}\xi)\\ \text{is analytic from }\mathbb{C}\text{ to }(B(\mathcal{H}),\|\ \|)\end{array}\right\}$$

Then $\mathcal{U}_0$ is a modular Hilbert algebra equivalent to $\mathcal{S}$.

Proof Let $\xi,\eta\in\mathcal{U}_0$, $\zeta\in\mathcal{S}$,

$$\Pi(\Delta^{\alpha}(\xi\eta))\zeta=\Delta^{\alpha}(\xi\eta)\zeta=(\Delta^{\alpha}\xi)(\Delta^{\alpha}\eta)\zeta=\Pi(\Delta^{\alpha}\xi)\Pi(\Delta^{\alpha}\eta)\zeta,$$

so that $\Pi(\Delta^{\alpha}(\xi\eta))=\Pi(\Delta^{\alpha}\xi)\Pi(\Delta^{\alpha}\eta)$ is analytic, i. e., $\xi\eta\in\mathcal{U}_0$. On the other hand, if $\xi\in\mathcal{U}_0$, $\Pi(\Delta^{\alpha}\xi^*)=(\Pi(\Delta^{-\bar\alpha}\xi))^*$, so that $\xi^*\in\mathcal{U}_0$. Hence $\mathcal{U}_0$ is a #-subalgebra of $\mathcal{U}$.

Now if $\xi\in\mathscr{S}$, by Proposition 1.3, $\xi_r\in\mathcal{U}$. Moreover

$$\Delta^{\alpha}\xi_r=\sqrt{\frac{r}{\pi}}\int_{-\infty}^{+\infty}e^{-r(t+i\alpha)^2}\Delta^{it}\xi dt,$$

$$\Pi(\Delta^{\alpha}\xi_r)=\sqrt{\frac{r}{\pi}}\int_{-\infty}^{+\infty}e^{-r(t+i\alpha)^2}\Delta^{it}\Pi(\xi)\Delta^{-it}dt,$$

so that $\xi_r\in\mathcal{U}_0$, i.e. $\mathcal{U}_a\subset\mathcal{U}_0$. This completes the proof.

Proposition 1.5

$$\mathcal{U}^2=\{\sum_i\xi_i\eta_i\,|\,\xi_i,\eta_i\in\mathcal{U}\}$$

is also a modular Hilbert algebra equivalent to $\mathscr{S}$.

Proof It is sufficient to prove that Δ^s is the closure of $\Delta^s|\mathcal{U}^2$, $\forall s\in\mathbb{R}$.

Fixed $s\in\mathbb{R}$, $\xi\in\mathfrak{D}(\Delta^s)$ and $\delta>0$. Because Δ^s is the closure of $\Delta^s|\mathcal{U}$, we have $\zeta\in\mathcal{U}$ such that

$$\|\zeta-\xi\|<\delta,\qquad \|\Delta^s\zeta-\Delta^s\xi\|<\delta$$

For arbitrary $\eta\in\mathscr{S}$, suppose

$$\tilde{\eta}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\Delta^{it}\eta dt\in\mathcal{U},$$

then

$$\Delta^s\tilde{\eta}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-(t+is)^2}\Delta^{it}\eta dt,$$

$$\Pi(\tilde{\eta})=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\Delta^{it}\Pi(\eta)\Delta^{-it}dt,\ \Pi(\Delta^s\tilde{\eta})=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-(t+is)^2}\Delta^{it}\Pi(\eta)\Delta^{-it}dt,$$

and

$$\|\Pi(\tilde{\eta})\zeta-\xi\|\leqslant\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\|\Pi(\eta)\Delta^{it}\zeta-\Delta^{it}\xi\|dt,$$

$$\|\Pi(\Delta^s\tilde{\eta})\Delta^s\zeta-\Delta^s\xi\|\leqslant\frac{e^{s^2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\|\Pi(\eta)\Delta^{it}\Delta^s\eta-\Delta^{it}\Delta^s\xi\|dt.$$

Suppose $\{t_k\}$ be a dense subset of $\mathbb{R}$, because $I\in\{\Pi(\mathscr{S})\}''$, hence for every n, there exists a $\eta_n\in\mathscr{S}$ such that

$$\|\Pi(\eta_n)\|\leqslant1,\qquad \|(\Pi(\eta_n)-I)\Delta^{it_k}\zeta'\|<\frac{1}{n},\qquad 1\leqslant k\leqslant n,$$

where $\zeta'=\zeta$ or $\Delta^s\zeta$. It is not difficult to prove that

$$\|(\Pi(\eta_n)-I)\Delta^{it}\zeta'\|\xrightarrow{n}0,\qquad \forall t\in\mathbb{R},$$

By the theorem of dominated convergence, we have

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\|\Pi(\eta_n)\Delta^{it}\zeta-\Delta^{it}\xi\|dt\xrightarrow{n}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\|\Delta^{it}\zeta-\Delta^{it}\xi\|dt=\|\zeta-\xi\|,$$

$$\frac{e^{s^2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-t^2}\|\Pi(\eta_n)\Delta^{it}\Delta^s\zeta-\Delta^{it}\Delta^s\xi\|dt$$

$$\xrightarrow{n}\frac{e^{s^2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-t^2}\|\Delta^{it}\Delta^s\zeta-\Delta^{it}\Delta^s\xi\|dt=\|\Delta^s\zeta-\Delta^s\xi\|e^{s^2},$$

when n is sufficiently large, let $\eta=\tilde{\eta}_n\in\mathcal{U}$, then

$$\|\Pi(\eta)\zeta-\xi\|<\delta,\qquad \|\Pi(\Delta^s\eta)\Delta^s\zeta-\Delta^s\xi\|<e^{s^2}\delta.$$

But $\Pi(\eta)\zeta=\eta\zeta\in\mathcal{U}^2$, $\Pi(\Delta^s\eta)\Delta^s\zeta=\Delta^s(\eta\zeta)$, therefore Δ^s is the closure of $\Delta^s|\mathcal{U}^2$. Q. E. D.

Lemma 1.6 $x\in B(\mathcal{H})$ is called analytic about $\{\Delta^{it}\}$, if there is an analytic map $x(\alpha)$ from $\mathbb{C}$ to $(B(\mathcal{H}),\|\ \|)$ such that

$$x(t)=\Delta^{it}x\Delta^{-it},\qquad \forall t\in\mathbb{R}.$$

Then $x\mathscr{D}(\Delta^\alpha)\subset\mathscr{D}(\Delta^\alpha)$, and $x(\alpha)\supset\Delta^{i\alpha}x\Delta^{-i\alpha}$, $\forall\alpha\in\mathbb{C}$.

Proof If $s>0$ and $\xi\in\mathscr{D}(\Delta^s)$, then $\Delta^{it}\xi(t\in\mathbb{R})$ can be extended to become a function $\xi(z)$ which is boundedly continuous in $-s\leqslant \mathrm{Im}z\leqslant 0$ and analytic in $-s<\mathrm{Im}z<0$. so is $\Delta^{it}x\xi=\Delta^{it}x\Delta^{-it}\cdot\Delta^{it}\xi$. Hence $x\xi\in\mathscr{D}(\Delta^s)$. Similarly $x\mathscr{D}(\Delta^s)\subset\mathscr{D}(\Delta^s)$ when $s<0$. Therefore $x\mathscr{D}(\Delta^\alpha)\subset\mathscr{D}(\Delta^\alpha)$, $\forall\alpha\in\mathbb{C}$.

If $\xi,\eta\in\mathscr{D}$, then

$$\langle\Delta^{i\alpha}x\Delta^{-i\alpha}\xi,\eta\rangle=\langle x(\Delta^{-i\alpha}\xi),\Delta^{-i\bar{\alpha}}\eta\rangle$$

is analytic on $\mathbb{C}$, therefore

$$\langle x(\alpha)\xi,\eta\rangle=\langle\Delta^{i\alpha}x\Delta^{-i\alpha}\xi,\eta\rangle\qquad\forall\alpha\in\mathbb{C}$$

and

$$x(\alpha)|\mathscr{D}=\Delta^{i\alpha}x\Delta^{-i\alpha}|\mathscr{D}\qquad\forall\alpha\in\mathbb{C}$$

Because $\mathscr{D}(\Delta^{i\alpha}x\Delta^{-i\alpha})=\mathscr{D}(\Delta^{-i\alpha})$ and $\Delta^{-i\alpha}$ is the closure of $\Delta^{-i\alpha}|\mathscr{D}$, so it is not difficult to prove that $x(\alpha)\supset\Delta^{i\alpha}x\Delta^{-i\alpha}$, $\forall\alpha\in\mathbb{C}$. Q. E. D.

Lemma 1.7 Let $\xi\in\mathscr{S}\cap\mathscr{D}$ and $\alpha\in\mathbb{C}$, then $\Delta^\alpha\xi\in\mathscr{S}$ if and only if there is an operator $A\in B(\mathcal{H})$ such that

$$A\supset\Delta^\alpha\Pi(\xi)\Delta^{-\alpha}.$$

However, in this case, $A=\Pi(\Delta^\alpha\xi)$.

Proof Let $\Delta^\alpha\xi\in\mathscr{S}$ and $\eta\in\mathcal{U}_0(\mathscr{S}')$, by lemma 1.6,

$$\Pi(\Delta^\alpha\xi)\eta=\Delta^\alpha\Delta^{-\alpha}\Pi'(\eta)\Delta^\alpha\xi=\Delta^\alpha\Pi'(\Delta^{-\alpha}\eta)\xi=\Delta^\alpha\Pi(\xi)\Delta^{-\alpha}\eta$$

Hence

$$\Pi(\Delta^\alpha\xi)\supset\Delta^\alpha\Pi(\xi)\Delta^{-\alpha}|\mathcal{U}_0(\mathscr{S}')$$

But by Proposition 1.4, $\Delta^{-\alpha}$ is the closure of $\Delta^{-\alpha}|\mathcal{U}_0(\mathscr{S}')$, so that $\Pi(\Delta^\alpha\xi)\supset\Delta^\alpha\Pi(\xi)\Delta^{-\alpha}$.

Now let $A\in B(\mathcal{H})$ and $A\supset\Delta^\alpha\Pi(\xi)\Delta^{-\alpha}$, $\eta\in\mathcal{U}_0(\mathscr{S}')$, then

$$\Pi(\Delta^\alpha\xi)\eta=\Pi'(\eta)\Delta^\alpha\xi=\Delta^\alpha\Pi'(\Delta^{-\alpha}\eta)\xi=\Delta^\alpha\Pi(\xi)\Delta^{-\alpha}\eta=A_\eta$$

when $\eta\in\mathscr{S}'$, take $\eta_n\in\mathcal{U}_0(\mathscr{S}')$ such that $\|\eta_n-\eta\|\xrightarrow{n}0$, then

$$\Pi(\Delta^\alpha\xi)\eta_n=A_{\eta_n}\xrightarrow{n}A_\eta$$

But $\Pi(\Delta^{\alpha}\xi)$ is a closed operator, hence

$$\|\Pi(\Delta^{\alpha}\xi)\eta\| = \|A_{\eta}\| \leqslant \|A\| \ \|\eta\| \qquad \forall \eta \in \mathscr{S}'$$

i. e., $\Delta^{\alpha}\xi$ is a left bounded element. Because $\Delta^{\alpha}\xi \in \mathscr{D}(\Delta^{\frac{1}{2}})$ also, therefore $\Delta^{\alpha}\xi \in \mathscr{S}'' = \mathscr{S}$. Q. E. D.

Proposition 1.8

$$\mathscr{U}_0 = \{\xi \mid \xi \in \mathscr{U}, \Pi(\xi) \text{ is analytic about } \{\Delta^{it}\}\}$$
$$= \{\xi \mid \xi \in \mathscr{S} \cap \mathscr{D}, \Pi(\xi) \text{ is analytic about } \{\Delta^{it}\}\}$$
$$= \left\{\xi \;\middle|\; \begin{array}{l} \text{there is a mapping } \xi(\alpha): \mathbb{C} \to \mathscr{S} \text{ such that} \\ \xi(t) = \Delta^{it}\xi, \ \forall t \in \mathbb{R} \text{ and } \alpha \to \Pi(\xi(\alpha)) \text{ is analytic} \\ \text{from } \mathbb{C} \text{ to } (B(\mathscr{H}), \|\ \|) \end{array}\right\}$$

Proof let $\xi \in \mathscr{S} \cap \mathscr{D}$ and $\Pi(\xi)$ is analytic about $\{\Delta^{it}\}$, by lemma 1.6, $\Pi(\xi)(\alpha) \supset \Delta^{i\alpha}\Pi(\xi)\Delta^{-i\alpha}$, further by lemma 1.7, $\Delta^{\alpha}\xi \in \mathscr{S}$, $\forall \alpha \in \mathbb{C}$, so that $\xi \in \mathscr{U}$.

Now if $\xi \in \mathscr{U}$, and $\Pi(\xi)$ is analytic about $\{\Delta^{it}\}$, by lemma 1.6 and 1.7, $\Pi(\xi)(\alpha) = \Pi(\Delta^{\alpha}\xi)$, $\forall \alpha \in \mathbb{C}$, so that $\alpha \to \Pi(\Delta^{\alpha}\xi)$ is analytic from $\mathbb{C}$ to $(B(\mathscr{H}), \|\ \|)$, i. e., $\xi \in \mathscr{U}_0$.

Now let the function $\alpha \to \xi(\alpha)$: $\mathbb{C} \to \mathscr{S}$ such that $\xi(t) = \Delta^{it}\xi$, $\forall t \in \mathbb{R}$, and $\alpha \to \Pi(\xi(\alpha))$ is analytic from $\mathbb{C}$ to $(B(\mathscr{H}), \|\ \|)$, we must prove $\xi \in \mathscr{D}$. Suppose $\alpha \in \mathbb{C}$, and $\eta, \zeta \in \mathscr{U}_0(\mathscr{S}')$, then

$$\langle \Pi(\xi(\alpha))\eta, \zeta\rangle = \langle \Pi'(\eta)\xi(\alpha), \zeta\rangle = \langle \xi(\alpha), \eta^{\flat}\zeta\rangle$$

By lemma 1.6, $\Pi(\xi(\alpha)) \supset \Delta^{i\alpha}\Pi(\xi)\Delta^{-i\alpha}$, so that

$$\langle \Pi(\xi(\alpha))\eta, \zeta\rangle = \langle \Delta^{i\alpha}\Pi(\xi)\Delta^{-i\alpha}\eta, \zeta\rangle = \langle \Pi'(\Delta^{-i\alpha}\eta)\xi, \Delta^{-i\bar{\alpha}}\zeta\rangle = \langle \xi, \Delta^{-i\bar{\alpha}}(\eta^{\flat}\zeta)\rangle$$

By Proposition 1.5, $\Delta^{-i\bar{\alpha}}$ is the closure of $\Delta^{-i\bar{\alpha}} \mid \mathscr{U}_0(\mathscr{S}')^2$, therefore $\xi \in \mathscr{D}(\Delta^{i\alpha})$ and $\xi(\alpha) = \Delta^{i\alpha}\xi$, $\forall \alpha \in \mathbb{C}$. Q. E. D.

§2 The modular Hilbert algebras which fail to satisfy the condition (Ⅷ)

Let $(\mathscr{U}, \#, \Delta(\alpha), \langle , \rangle)$ satisfy the conditions (Ⅰ)—(Ⅶ) of modular Hilbert algebras, but except the condition (Ⅷ) ([1]), $\mathscr{H}$ be the completion of $(\mathscr{U}, \langle , \rangle)$.

By the conditions (Ⅴ) and (Ⅶ), $\{\Delta(it)\mid_{t\in\mathbb{R}}\}$ can be uniquely extended to become a strongly continuous group of unitary operators $\{U(t)\mid_{t\in\mathbb{R}}\}$ in $\mathscr{H}$. Then by Stone's theorem, there is an unique positive self-adjoint operator $\tilde{\Delta}$ in $\mathscr{H}$ such that

$$U(t) = \tilde{\Delta}^{it}, \qquad \forall t \in \mathbb{R}$$

From the condition (Ⅶ), we have

$$\mathscr{U} \subset \tilde{\mathscr{D}} = \bigcap_{\alpha\in\mathbb{C}} \mathscr{D}(\tilde{\Delta}^{\alpha}), \qquad \tilde{\Delta}^{\alpha} \supset \Delta(\alpha) \qquad \forall \alpha \in \mathbb{C}$$

Suppose
$$\tilde{J}\xi = \Delta\left(\frac{1}{2}\right)\xi^{*} = \Delta^{\frac{1}{2}}\xi^{*} \qquad \forall \xi \in \mathscr{U}$$

Then $\tilde{J}$ can be uniquely extended to a bounded conjugate linear operator in $\mathscr{H}$ (still denoted by $\tilde{J}$) such that

$$\tilde{J}^2 = I, \quad \langle \tilde{J}\xi, \tilde{J}\eta\rangle = \langle \eta, \xi\rangle, \qquad \forall \xi, \eta \in \mathscr{H}$$

Because

$$\tilde{J}\tilde{\Delta}^{\frac{1}{2}}\xi = \xi^*, \qquad \forall \xi \in \mathcal{U}$$

hence the operator $\#$ (with domain $\mathcal{U}$) has a closed extension in $\mathscr{H}$. Therefore ($\mathcal{U}$, $\#$, $<,>$) is also a generalized Hilbert algebra, let its unitary involution and modular operator be J and Δ.

Lemma 2.1 $J = \tilde{J}$, if and only if $\Delta = \tilde{\Delta}$.

Proof Let S be the closure of the operator $\#$ in $\mathscr{H}$.

If $J = \tilde{J}$, by $\tilde{J}\tilde{\Delta}^{\frac{1}{2}} \supset S = J\Delta^{\frac{1}{2}}$, so that $\tilde{\Delta}^{\frac{1}{2}} \supset \Delta^{\frac{1}{2}}$. But $\tilde{\Delta}^{\frac{1}{2}}$ and $\Delta^{\frac{1}{2}}$ are all self-adjoint, hence $\Delta = \tilde{\Delta}$. Conversely let $\Delta = \tilde{\Delta}$, then $\mathscr{D}(S) = \mathscr{D}(\tilde{\Delta}^{\frac{1}{2}}) = \mathscr{D}(\Delta^{\frac{1}{2}})$. By $\tilde{J}\tilde{\Delta}^{\frac{1}{2}} \supset S$, so that $S = \tilde{J}\tilde{\Delta}^{\frac{1}{2}}$. Now by the uniqueness of polar decomposition, $J = \tilde{J}$. Q. E. D.

Lemma 2.2 Let

$$K = \{\xi \mid \xi = \xi^* \in \mathcal{U}\}$$

Then $\{\Delta^{it}|_{t\in\mathbb{R}}\}$ is the unique strongly continuous group of unitary operators in $\mathscr{H}$ such that $\Delta^{it}K \subset K (\forall t \in \mathbb{R})$ and for arbitrary $\xi, \eta \in K$, there is a (K. M. S.) function f(z) which is boundedly continuous in $0 \leqslant \mathrm{Im}z \leqslant 1$ and analytic in $0 < \mathrm{Im}z < 1$ and satisfies

$$f(t) = \langle \eta, \Delta^{it}\xi\rangle = \overline{f(t+i)}, \qquad \forall t \in \mathbb{R}$$

Proof Because S is the closure of $S|\mathcal{U}$ and $\Delta^{it}\mathcal{U}'' = \mathcal{U}''$, so that $\Delta^{it}K \subset K$, $\forall t \in \mathbb{R}$.

Let $\xi, \eta \in K \subset \mathscr{D}(S) = \mathscr{D}(\Delta^{\frac{1}{2}})$, suppose

$$\xi_n = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} e^{-nt^2}\Delta^{it}\xi dt, \qquad f_n(z) = \langle \eta, \Delta^{iz}\xi_n\rangle$$

then $\|\xi_n - \xi\| \xrightarrow{n} 0$, $f_n(z)$ is analytic on $\mathbb{C}$ and bounded in $0 \leqslant \mathrm{Im}z \leqslant 1$ (because $\xi_n \in \mathscr{D}$), and

$$f_n(t+i) = \langle \eta, \Delta^{it}\Delta\xi_n\rangle = \langle \Delta^{\frac{1}{2}}\eta, \Delta^{it}\Delta^{\frac{1}{2}}\xi_n\rangle = \langle \Delta^{it}S\xi_n, S\eta\rangle = \langle \Delta^{it}\xi_n, \eta\rangle = \overline{f_n(t)}, \quad \forall t \in \mathbb{R}$$

Now by the principle of Maximum norm, $f_n(z) \xrightarrow{n} f(z)$ uniformly in $0 \leqslant \mathrm{Im}z \leqslant 1$, and $f(z)$ is the K. M. S. function of ξ, η.

The uniqueness, see [3]. Q. E. D.

Lemma 2.3 $\Delta = \tilde{\Delta}$, in particular, $\left(1 + \Delta\left(\frac{1}{2}\right)\right)\mathcal{U}$ is dense in $\mathscr{H}$.

Proof By Lemma 2.2, it is sufficient to prove that $U(t)K \subset K (\forall t \in \mathbb{R})$ and $\{U(t)\}$ satisfies the K. M. S. condition about K.

Let $\xi = \xi^* \in \mathcal{U}$, then

$$(U(t)\xi)^* = (\Delta(it)\xi)^* = \Delta(-\overline{it})\xi^* = \Delta(it)\xi = U(t)\xi \in \mathcal{U}$$

so that $U(t)K\subset K$, $\forall t\in\mathbb{R}$.

Let $\xi=\xi^*$, $\eta=\eta^*\in\mathcal{U}$, then

$$f(z)=\langle\eta,\tilde{\Delta}^{i\bar{z}}\xi\rangle=\langle\eta,\Delta(i\bar{z})\xi\rangle=\langle\Delta(-iz)\eta,\xi\rangle$$

is analytic on $\mathbb{C}$, and bounded in $0\leqslant \mathrm{Im}z\leqslant 1$ (because $\xi\in\bigcap_{a\in\mathbb{C}}\mathscr{D}(\tilde{\Delta}^a)$), and

$$f(t+i)=\langle\Delta(-it+1)\eta,\xi\rangle=\langle\Delta(1)\eta,\Delta(it)\xi\rangle$$
$$=\langle(\Delta(it)\xi)^*,\eta^*\rangle=\langle\Delta(it)\xi,\eta\rangle=\langle U(t)\xi,\eta\rangle=\overline{f(t)},\quad\forall t\in\mathbb{R}$$

so that $f(z)$ is the K. M. S. function of ξ,η.

Now if $\xi,\eta\in K$, take $\xi_n=\xi_n^*,\eta_n=\eta_n^*\in\mathcal{U}$, such that $\|\xi_n-\xi\|\xrightarrow{n}0$, $\|\eta_n-\eta\|\xrightarrow{n}0$. By $f_n(z)=\langle\eta_n,\Delta(i\bar{z})\xi_n\rangle$ is the K. M. S. function of ξ_n, η_n, and the principle of Maximum norm, we have

$$|f_n(z)-f_m(z)|\leqslant\sup_{t\in\mathbb{R}}|f_n(t)-f_m(t)|\leqslant\|\xi_n-\xi_m\|\ \|\eta_m\|+\|\eta_n-\eta_m\|\ \|\xi_n\|\xrightarrow{n,m}0$$

uniformly for $0\leqslant\mathrm{Im}z\leqslant 1$. Therefore $f_n(z)\xrightarrow{n}f(z)$ and $f(z)$ is the K. M. S. function of ξ,η. Q. E. D.

Proposition 2.4 $(\mathcal{U},\#,\Delta(a),<,>)$ can be extended to become a modular Hilbert algebra in $\mathscr{H}$.

Proof By Lemma 2.1, 2.3, $J=\tilde{J},\Delta=\tilde{\Delta}$, then the maximum modular Hilbert algebra equivalent to $\mathcal{U}''$ is an extension of $\mathcal{U}$. Q. E. D.

§3 #-two-sided ideals of a modular Hilbert algebra

Let $(\mathcal{U},\#,\Delta(a),<,>)$ be a modular Hilbert algebra, $\mathscr{H}$ be the completion of $(\mathcal{U},<,>)$, Δ be its modular operator.

Lemma 3.1 Let $\mathscr{S}$ be a #-subalgebra of $\mathcal{U}$, $\Delta(a)\mathscr{S}\subset\mathscr{S}$, $\forall a\in\mathbb{C}$, and K be the closed subspace of $\mathscr{H}$ generated by $\mathscr{S}$. Then $(\mathscr{S},\#,\Delta(a),<,>)$ is also a modular Hilbert algebra, if and only if, for every $s\in\mathbb{R}$ and $\xi\in K\cap\mathscr{D}(\Delta^s)$, there exists a sequence $\{\xi_n\}\subset\mathscr{S}$ such that

$$\|\xi_n-\xi\|\xrightarrow{n}0,\quad\|\Delta(s)\xi_n-\Delta^s\xi\|\xrightarrow{n}0$$

Proof According to §2, $\mathscr{S}$ is also a generalized Hilbert algebra in K, let $\Delta_{\mathscr{S}}$ be its modular operator.

Because $\Delta(it)\mathscr{S}\subset\mathscr{S}$, so that

$$\Delta^{it}|_K=(\Delta_{\mathscr{S}})^{it},\qquad\forall t\in\mathbb{R}$$

Therefore $\Delta^a|_K=(\Delta_{\mathscr{S}})^a$, $\forall a\in\mathbb{C}$.

It is obvious that $(\mathscr{S},\#,\Delta(a),<,>)$ is also a modular Hilbert algebra, if and only if, $(1+\Delta_{\mathscr{S}}^s)\mathscr{S}$ is dense in K, $\forall s\in\mathbb{R}$, i. e., $\Delta_{\mathscr{S}}^s=\Delta^s|_K$ is the closure of $\Delta(s)|\mathscr{S}$. This completes the proof.

Proposition 3.2 Let $\mathscr{I}$ be a #-two-sided ideal of $\mathcal{U}$, and $\Delta(a)\mathscr{I}\subset\mathscr{I}$, $\forall a\in\mathbb{C}$, then $(\mathscr{I},\#,\Delta(a),<,>)$ is also a modular Hilbert algebra.

Proof Let K be the closed subspace generated by $\mathscr{I}$. For any fixed $s\in\mathbb{R}$, $\xi\in K\cap\mathscr{D}(\Delta^s)$ and $\delta>0$, there exists $\zeta\in\mathscr{U}$ such that

$$\|\zeta-\xi\|<\delta,\quad \|\Delta^s\zeta-\Delta^s\xi\|<\delta$$

For any $\eta\in\mathscr{I}$. suppose

$$\tilde{\eta}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\Delta^{it}\eta dt$$

According to the proof of Proposition 1.5, we have

$$\|\Pi(\tilde{\eta})\zeta-\xi\|\leqslant\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\|\Pi(\eta)\Delta^{it}\zeta-\Delta^{it}\xi\|dt$$

$$\|\Pi(\Delta^s\tilde{\eta})\Delta^s\zeta-\Delta^s\xi\|\leqslant\frac{e^{s^2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\|\Pi(\eta)\Delta^{it}\Delta^s\zeta-\Delta^{it}\Delta^s\xi\|dt$$

Suppose $\{t_k\}$ be a dense subset of $\mathbb{R}$, Q be the orthogonal projection from $\mathscr{H}$ onto $[\Pi(\mathscr{I})\mathscr{H}]$, then $Q\in\overline{\Pi(\mathscr{I})}''$. By the density theorem, for every n, there exists $\eta_n\in\mathscr{I}$ such that

$$\|\Pi(\eta_n)\|\leqslant 1,\quad \|[\Pi(\eta_n)-Q]\Delta^{it_k}\zeta'\|<\frac{1}{n},\quad 1\leqslant K\leqslant n$$

where $\zeta'=\zeta$ or $\Delta^s\zeta$, Then

$$\|[\Pi(\eta_n)-Q]\Delta^{it}\zeta'\|\xrightarrow{n}0,\quad \forall t\in\mathbb{R}$$

Hence we have

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\|\Pi(\eta_n)\Delta^{it}\zeta-\Delta^{it}\xi\|dt\xrightarrow{n}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\|Q\Delta^{it}\zeta-\Delta^{it}\zeta\|dt$$

$$\frac{e^{s^2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\|\Pi(\eta_n)\Delta^{it}\Delta^s\zeta-\Delta^{it}\Delta^s\xi\|dt\xrightarrow{n}\frac{e^{s^2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\|Q\Delta^{it}\Delta^s\zeta-\Delta^{it}\Delta^s\xi\|dt$$

When n is sufficiently large, let $\eta=\eta_n\in\mathscr{I}$, then

$$\|\Pi(\tilde{\eta})\zeta-\xi\|\leqslant\delta+\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\|Q\Delta^{it}\zeta-\Delta^{it}\xi\|dt$$

$$\|\Pi(\Delta^s\tilde{\eta})\Delta^s\zeta-\Delta^s\xi\|\leqslant\delta+\frac{e^{s^2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\|Q\Delta^{it}\Delta^s\zeta-\Delta^{it}\Delta^s\xi\|dt$$

$\mathscr{I}$ is also a generalized Hilbert algebra, hence

$$K=\overline{\mathscr{I}}=\overline{\mathscr{I}^2}=[\Pi(\mathscr{I})\mathscr{I}]\subset Q\mathscr{H}$$

Because $\Delta(\alpha)\mathscr{I}\subset\mathscr{I}$, hence $Q\sim\Delta^\alpha$, $\forall\alpha\in\mathbb{C}$. By $\xi\in K$, therefore $Q\Delta^{it}\xi=\Delta^{it}\xi$, $Q\Delta^{it}\Delta^s\xi=\Delta^{it}\Delta^s\xi$, $\forall t\in\mathbb{R}$ and

$$\|\Pi(\tilde{\eta})\zeta-\xi\|<2\delta,\quad \|\Pi(\Delta^s\tilde{\eta})\Delta^s\zeta-\Delta^s\xi\|<(1+e^{s^2})\delta$$

Because of

$$\tilde{\eta}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\Delta^{it}\eta dt=\lim_{\mathscr{P}}\sum_j(t_j-t_{j-1})e^{-t_j^2}\Delta^{it_j}\eta$$

and

$$\Delta^s\tilde{\eta}=\widetilde{\Delta^s\eta}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-t^2}\Delta^{it}\Delta^s\eta dt$$

so that we can take $\eta'=\sum_j(t_j-t_{j-1})e^{-t_j^2}\Delta^{it_j}\eta\in\mathscr{I}$ such that

$$\|\Pi(\eta')\zeta-\Pi(\tilde{\eta})\zeta\|\leqslant\|\Pi'(\zeta)\|\ \|\eta'-\tilde{\eta}\|\leqslant\delta$$
$$\|\Pi(\Delta^s\tilde{\eta})\Delta^s\zeta-\Pi(\Delta^s\eta')\Delta^s\zeta\|\leqslant\|\Pi'(\Delta^s\zeta)\|\ \|\Delta^s\eta'-\Delta^s\tilde{\eta}\|\leqslant\delta$$

(because $\zeta,\Delta^s\zeta\in\mathscr{U}$ and $\mathscr{U}\subset\mathscr{U}''\cap\mathscr{U}'$). Therefore

$$\|\eta'\zeta-\xi\|<3\delta,\quad \|\Delta^s(\eta'\zeta)-\Delta^s\xi\|<(2+e^{s^2})\delta$$

Now $\mathscr{I}$ is a ideal of $\mathscr{U}$ and $\eta'\in\mathscr{I}$, so that $\eta'\zeta\in\mathscr{I}$. By lemma 3.1, we complete the proof.

Proposition 3.3 Let $(\mathscr{U},\#,\Delta(\alpha),<,>)$ be a modular Hilbert algebra, $\mathscr{L}(\mathscr{U})$ be its left von Neumann algebra, Θ be a σ-closed two-sided ideal of $\mathscr{L}(\mathscr{U})$. If

$$\Pi(\mathscr{I})=\Pi(\mathscr{U})\cap\Theta$$

then $\mathscr{I}$ is a $\#$-two-sided ideal of $\mathscr{U}$, and $\Delta(\alpha)\mathscr{I}\subset\mathscr{I}$, $\forall\alpha\in\mathbb{C}$.

Proof It is obvious that $\mathscr{I}$ is a $\#$-two-sided ideal of $\mathscr{U}$.

Let $\Theta=\mathscr{L}(\mathscr{U})z$, where z is a central projection of $\mathscr{L}(\mathscr{U})$. If $\alpha\in\mathbb{C}$ and $\xi\in\mathscr{I}$, it is sufficient to prove that $\Pi(\Delta^\alpha\xi)z=\Pi(\Delta^\alpha\xi)$.

By [4], $\Delta^\lambda z\supset z\Delta^\lambda$, $\forall\lambda\in\mathbb{C}$. By lemma 1.6

$$\Pi(\Delta^\alpha\xi)\eta=\Delta^\alpha\Pi(\xi)\Delta^{-\alpha}\eta,\qquad\forall\eta\in\mathscr{D}(\Delta^{-\alpha})$$

Therefore

$$\Pi(\Delta^\alpha\xi)z\eta=\Delta^\alpha\Pi(\xi)\Delta^{-\alpha}z\eta=\Delta^\alpha\Pi(\xi)z\Delta^{-\alpha}\eta=\Delta^\alpha\Pi(\xi)\Delta^{-\alpha}\eta=\Pi(\Delta^\alpha\xi)\eta\qquad\forall\eta\in\mathscr{D}(\Delta^{-\alpha}),$$

further $\Pi(\Delta^\alpha\xi)z=\Pi(\Delta^\alpha\xi)$. Q. E. D.

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