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Stability of Solution of a Scalar Differential Equation*

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Lyapunov's second method in fact is that which may be described as follows: applying the comparison principle to the V-function, we may render the stability of the solution of a vector differential equation to that of the scalar differential equation

$$\frac{du}{dt}=\omega(t,u),\ \ \omega(t,0)\equiv 0,\ \ 0\leqslant u\leqslant\rho(t),\ \ t\in\mathscr{T} \tag{1}$$

[cf. C. Corduneanu, 1960]. Unfortunately, the stability of the solutions of scalar differential equations had not been well-discussed. In literatures, ρ is assumed to be a positive constant and $\omega(t,u)$ is nondecreasing for a fixed t. In this paper the author dropps these two limitations and obtains several sufficient conditions for the stability and unstability of the solutions.

Consider the d. e. (1). Let $\omega\in C[D,R_+]$, where $D=\{(t,u)\in R^2: 0\leqslant u\leqslant\rho(t), t\in\mathscr{T}\}$, $\rho(t)$ is continuous on $\mathscr{T}$. Assumed that the solutions of d. e. (1) is unique and continuous with respect to initial values.

Theorem 1 We consider the scalar d. e.,

$$\frac{du}{dt}=F(t,u)u+f(t,u). \tag{2}$$

Suppose that: 1) $F,f\in C[D,R_+]$, $f(t,0)\equiv 0$ and $\int_{t_0}^{t}F(\tau,u(\tau))d\tau\leqslant\int_{t_0}^{t}F(\tau,\rho(\tau))d\tau$, $\int_{t_0}^{t}f(\tau,u(\tau))d\tau\leqslant\int_{t_0}^{t}f(\tau,\rho(\tau))d\tau$, for $0\leqslant u\leqslant\rho(t)$; 2) $\int_{t_0}^{\infty}F(t,\rho(t))dt<+\infty$, $\int_{t_0}^{\infty}f(t,\rho(t))dt<+\infty$ for $t\in\mathscr{T}$. Then the trivial solution $u=0$ of (2) is stable. Assume that the hypothesis of theorem 1 holds expect that $\int_{t_0}^{\infty}F(t,\rho(t))dt<+\infty$ is replaced by $\int_{t_0}^{\infty}F(t,\rho(t))dt=-\infty$. Then the solution $u=0$ is asymptotically stable.

Proof By assumption 2), $\exists G>0$, $\ni\cdot\int_{t_0}^{t}F(\tau,\rho(\tau))d\tau<\ln G$, $t>t_0$. For $\forall\varepsilon>0$, $\varepsilon<\rho(t)$, $\exists t_1>t_0\ni\cdot\int_{t_1}^{t}f(\tau,\rho(\tau))d\tau<\varepsilon/2G$, $t>t_1$. Choose $u_1<\varepsilon/2G$, We shall prove $u(t,t_1,u_1)<\varepsilon$, whenever $t>t_1$. For otherwise, there would exist $t_2<t_3\ni\cdot u(t_2)=\frac{\varepsilon}{2G}$

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$\leqslant u(t)\leqslant\varepsilon=u(t_3)<\rho(t)$, for $t_2\leqslant t\leqslant t_3$. We obtain an inequality

$$u(t)\leqslant\frac{\varepsilon}{G}\exp\int_{t_2}^{t}F(\tau,\rho(\tau))d\tau<\varepsilon,\ \text{and then}\ u(t_3)=\varepsilon<\varepsilon. \qquad (3)$$

This contradication proves the solution $u=0$ is stable.

Finally, under the assumptions $\int_{t_0}^{\infty}F(t,\rho(t))dt=-\infty$, we prove that the solution $u=0$ is asymptotically stable. By (2), $u(t)\leqslant 2u(t_2)+\int_{t_2}^{t}F(\tau,u(\tau))\,u(\tau)d\tau$, where $\int_{t_2}^{t}f(\tau,\rho(\tau))d\tau<u(t_2)$. Thus, $u(t)\leqslant 2u(t_2)\exp\int_{t_2}^{t}F(\tau,\rho(\tau))d\tau\to 0$, $t\to\infty$.

If $\int_{t_0}^{\infty}F(t,\rho(t))\,dt,\int_{t_0}^{\infty}f(t,\rho(t))dt$ is uniformly convergent for t_0, then solution $u=0$ is uniformly stable.

Theorem 2 Consider the d. e.

$$\frac{du}{dt}=F(t,u)g(u)+f(t,u). \qquad (4)$$

Suppose that: 1) Let $g(u)$ be continuous, positive and nondecreasing with respect to $u\in[0,\rho_0]$, $g(0)=0$, $G(u)=\int_{0+}^{u}\frac{ds}{g(s)}$ be convergent. 2) There exist positive number $H\leqslant\rho_0\ \ni\cdot\int_{t_0}^{\infty}f(t,u(t))dt\leqslant\int_{t_0}^{\infty}f(t,H)dt<+\infty$ and $\int_{t_0}^{\infty}F(t,u(t))dt\leqslant\int_{t_0}^{\infty}F(t,H)\,dt<+\infty$ for $u\leqslant H$. Then, the solution $u=0$ of (4) is stable. Assume that the hypothesis 2) holds expect that $\int_{t_0}^{\infty}F(t,H)dt<+\infty$ is replaced by $\int_{t_0}^{\infty}F(t,H)dt=-\infty$. Then the solution $u=0$ is asymptotically stable.

Proof Let $G(0)=0$ and $G(u)$ be continuous in $u=0$. Choose positive $\eta<H$, $\ni\cdot G(\eta)\leqslant G(H)/2$. By the hypothesis 2), $\exists t_1>t_0\ni\cdot\int_{t_1}^{t}f(\tau,H)d\tau<\eta/2$ and $\int_{t_1}^{t}F(\tau,H)d\tau<G(H)/2$. Choose $u_1<\eta/2$. At first, that proves $u(t,t_1,u_1)<H$, for $t>t_1$. For otherwise there would exist $t_1>t_0\ni\cdot\int_{t_1}^{t}f(\tau,H)d\tau<\eta/2$ and $\int_{t_1}^{t}F(\tau,H)d\tau<G(H)/2$. If $u_1<\eta/2$, prove that $u(t,t_1,u_1)<H$ for $t>t_1$. Otherwise, $\exists t_1<t_2<t_3\ni\cdot u(t_2)=\eta/2\leqslant u(t)\leqslant H=u(t_3)$, By (4) and Bihari's inequalities, we obtain

$$G(H)=G(u(t_3))<G(\eta)+G(H)/2<G(H),$$

where $G(u)=\int_{u_0}^{u}\frac{ds}{g(s)}$. This contradiction proves the solution $u=0$ is stable.

Finally, under the assumptions $\int_{t_0}^{\infty}F(t,\rho(t))dt=-\infty$, applying Bihari's inequalities we can prove easily that the solution $u=0$ is asymtotically stable.

Lemma Suppose that: 1) $u,\varphi\in C[\mathscr{T},R_+]$, where R_+ denotes the nonnegative real line; 2) $g\in C[R_+,R_+]$, $g(u)$ monotone in u and $g(0)=0$ and $kg(u)\leqslant g(ku)$, $k>0$, 3) $f(t)$ monotone in t and $f(t)>0$, $t\in\mathscr{T}$. Assume that $u(t)\leqslant f(t)+\int_{t_0}^{t}\varphi(\tau)g(u(\tau))d\tau$.

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Then $u(t)\leqslant f(t)G^{-1}[G(1)+\int_0^t \varphi(\tau)d\tau]$, $t>0$, where G^{-1} is the inverse function of G and $G(1)+\int_0^t \varphi(\tau)d\tau \in \mathrm{Dom}(G^{-1})$.

The last two theorem can be obtained as consequences of the preceding lemma.

Theorem 3 Suppose that: 1) $g(u)$ is monotonic nondecreasing and continuous for u, $g(0)=0$ and $kg(u)\leqslant g(ku), k>0$; 2) the hypothesis 2) of theorem 2 holds. Then the solution $u=0$ of (4) is stable. Assume that the hypothesis 2) $\int_{t_0}^t F(t,H)dt < +\infty$ is replaced by $\int_{t_0}^{\infty} F(t, H)dt = -\infty$. Then the solution $u=0$ is asymptotically stable.

Theorem 4 Consider d. e. (4). Suppose that: 1) $g(0)=0$, g is continuous non-negative and monotonic nondecreasing in u; 2) for sufficiently small b_1, b_2, $0<b_1<b_2 \ni \cdot \int_{t_0}^t F(\tau,b_1)d\tau \leqslant \int_{t_0}^t F(\tau,b_2)d\tau$, $\int_{t_0}^t f(\tau,b_1)d\tau \leqslant \int_{t_0}^t f(\tau,b_2)d\tau$. 3) $\int_{t_0}^{\infty} F(t,b)\,dt < +\infty$, $\int_{t_0}^{\infty} f(t,b)dt < +\infty$ for small $b>0$. Then the solution $u=0$ of (4) is stable. Assume that the hypothesis of theorem holds expect that $\int_{t_0}^t F(\tau,b)d\tau < +\infty$ is replaced by $\int_{t_0}^{\infty} F(\tau,b)d\tau = -\infty$. Then $u=0$ is asymptotically stable.

Theorem 5 Consider $\frac{du}{dt}=F(t,u)$, $t\in\mathscr{T}$, $0\leqslant u\leqslant \rho_0$. Suppose that: 1) $\exists H\leqslant\rho_0 \ni \cdot \int_{t_0}^t F(\tau,u(\tau))d\tau \geqslant \int_{t_0}^t F(\tau,H)d\tau$; 2) $\overline{\lim_{t\to\infty}} \int_{t_0}^t F(\tau,H)d\tau = +\infty$. Then $u=0$ is unstable.

This theorem is the extension of unstable theorem of Persidskij.

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