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On the Diophantine Equation $\sum_{k=1}^{m} k^n = (m+1)^n$*

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P. Erdös has conjectured [1] that the Diophantine equation

$$1^n + 2^n + \cdots + m^n = (m+1)^n \tag{1}$$

has no positive integer solutions except that $n=1$, $m=2$. It is true when $m \leqslant 10^{10^6}$ [3]. A generalized form of (1) has been investigated in [1] [2], and various results have been obtained, especially the conjecture is true when $n(\geqslant 5)$ is odd. In this short report, we outline the proof of the following results.

Theorem 1. Let n, m satisfy (1) and $n>1$, then we have the canonical forms:

(i) $m+1 = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, $p_j - 1 \nmid n$, $\sum_{k=1}^{p_j^{\alpha_j}} k^n \equiv 0 \pmod{p_j^{\alpha_j}}$, $(1 \leqslant j \leqslant s)$;

(ii) $m = p_1^{(1)} \cdots p_{s_1}^{(1)} = p_j^{(1)} m_j^{(1)}$, $p_1 = 2$, $p_j^{(1)} - 1 \mid n$, $p_j^{(1)} \mid m_j^{(1)} - 1$, $(1 \leqslant j \leqslant s_1)$;

(iii) $m+2 = 2^2 p_2^{(2)} \cdots p_{s_2}^{(2)} = p_j^{(2)} m_j^{(2)}$, $p_j^{(2)} - 1 \mid n$, $p_j^{(2)} \mid m_j^{(2)} + 2$, $(2 \leqslant j \leqslant s_2)$;

(iv) $2m+1 = p_1^{(3)} \cdots p_{s_3}^{(3)} = p_j^{(3)} m_j^{(3)}$, $p_j^{(3)} - 1 \mid n$, $p_j^{(3)} \mid m_j^{(3)} + 2$, $(1 \leqslant j \leqslant s_3)$;

(v) $2m+3 = p_1^{(4)} \cdots p_{s_4}^{(4)} = p_j^{(4)} m_j^{(4)}$, $p_j^{(4)} - 1 \mid n$, $p_j^{(4)} \mid m_j^{(4)} + 4$, $(1 \leqslant j \leqslant s_4)$

and that the conjecture is true when $m \leqslant (10^{10^6})^2$.

Theorem 2. The equation (1) has no positive integer solutions if one of the following conditions holds:

(i) $2 \| n$, $m \not\equiv 4 \pmod 5$;

(ii) $2 \| n$, $m \equiv 4 \pmod 5$ and $5 \nmid n$;

(iii) $2^\alpha \| n$, $m \not\equiv 2 \pmod{2^{\alpha+3}}$;

(iv) $2^\alpha \| n$, $m \equiv 2 \pmod{2^\beta}$, $\beta \geqslant \alpha + 4$.

It should be clear that all letters we used stand for positive integers.

In order to prove our theorems, we need the help of some lemmas.

Lemma 1. Let p be an odd prime, then n is not divided by p-1 if and only if

$$\sum_{k=1}^{p^\alpha} k^n \equiv 0 \pmod{p^\alpha}. \tag{2}$$

Lemma 2. If $n \equiv 0 \pmod 2$, then $\sum_{k=1}^{2^\alpha} k^n \equiv 2^{\alpha-1} \pmod{2^\alpha}$.

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Lemma 3. $2^\alpha \| Z$ is equivalent to $2^{\alpha+3} \| 3^z - 1$.

Lemma 4. Let n m satisfy (1) and $2^\alpha \| n$, then $n \geqslant \alpha + 3$ and $2^{\alpha+3} \| m-2$.

Lemma 1 is a well-known result when $\alpha = 1$ [4]. Suppose we have had the required result for $\alpha - 1$, and we wish to establish it for $\alpha > 1$. Noticing

$$\sum_{k=1}^{p^\alpha} k^n \equiv \sum_{l=0}^{p-1} \sum_{k=1}^{p^{\alpha-1}} (lp^{\alpha-1}+k)^n \equiv p \sum_{k=1}^{p^{\alpha-1}} k^n + n \cdot \frac{p-1}{2} \cdot p^\alpha \sum_{k=1}^{p^{\alpha-1}} k^{n-1} \pmod{p^\alpha}$$

and the second $\equiv 0 \pmod{p^\alpha}$, we immediatly get the sufficiency by the inductive assumption, Conversely, let g be a primitive root of p^α, we have $1, 2, \cdots, p^\alpha$ and $g \cdot 1, g \cdot 2, \cdots, g \cdot p^\alpha$ both are complete residue sets mod p^α. Thus, adding up modulo p^α, the fact that g is also a primitive root of p shows that (2) must hold.

A direct corollary of Lemma 1 is that the conjecture is true when n is odd.

Lemma 2—3 are easily checked by induction on the number α.

To prove Lemma 4, we first use Theorem 1 (its proof is independent of Lemma 4) and then use Lemma 2—3. Therefore the assertion follows. Now we can turn attention to our theorems.

Proof of Theorem 1. The proof of (i)— (v) can be carry out in turn by means of Lemma 1, which is omitted here. For the last conclusion, by the mentioned result in [3] we can, from (ii)—(v), deduce that

$$\sum_{j=1}^{q} \frac{1}{p_j} \leqslant 3.166666666 \text{ implies } m \geqslant \left(\frac{2}{3} \prod_{j=1}^{q-1} p_j\right)^{\frac{1}{4}},$$

where p_j is the jth prime. Using the datum given in [5][6], a few practical calculation shows that we may take $q = \pi(2 \times 10^7) = 1270607$ [6], so we have the bound.

Proof of Theorem 2. (i) by the (ii) (v) and (iv) of Theorem 1.

(ii) by Lemma 1 and induction to prove $\sum_{k=1}^{5^\alpha} k^n \not\equiv 0 \pmod{5^{\alpha+1}}$.

(iii) (iv) can be proved by Lemma 4.

References

[1] Ke Zhao and Sun Qi, On the Diophantine Equation $x^n + (x+1)^n + \cdots + (x+h)^n = (x+h+1)^n$, *Sichuan Daxue Xuebao* 2 (1962), 9—18. (in Chinese).

[2] Ke Zhao, Sun Qi and Zhou Zhao-nan, ——————, *Sichuan Daxue Xuebao* 2—3 (1978), 19—24. (in Chinese).

[3] Moser, L., On the diophantine equation $1^n + 2^n + \cdots + (m-1)^n = m^n$, *Scripta Math.* 19 (1953), 84—88.

[4] Hardy, G.H. and Wright, E. M., An introduction to the theory of numbers (fifth edition), Oxford (1981), 91.

[5] Hua Luo-geng, An introduction to number theory (fifth impression), *Kexue Chubanshe*(1979), 87. (in Chinese).

[6] Zagier, D., Die ersten 50 Millionen Primzahlen, Elemente der Mathematik Beihefle, № 15, Birkhaüer Verleg, Basel—Stutlgart, (1977) 24.