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On Univalent Functions with Negative and Missing Coefficients*

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Abstract Let $P_k(A,B)$ be the class of functions $f(z)=z-\sum_{n=k}^{\infty}|a_n|z^n$, $k\geqslant 2$, regular in the unit disc $U=\{z:|z|<1\}$ and satisfying $|[f'(z)-1]/[A-Bf'(z)]|<1$ for $z\in U$, where $-1\leqslant B<A\leqslant 1$. In this paper we obtain coefficient estimate, distortion and closure theorems and radius of convexity for the class $P_k(A,B)$ under the assumption $-1\leqslant B\leqslant 0$. We also obtain class preserving integral operators of the form

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt,\quad c>-1$$

for the class $P_k(A,B)$. Conversely when $F\in P_k(A,B)$, radius of univalence of f has been determined.

1. Introduction

Let S be the class of functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ regular in the unit disc $U=\{z:|z|<1\}$. For $-1\leqslant B<A\leqslant 1$, let $P^*(A,B)$ be the class of those functions f of S for which $f'(z)$ is subordinate to $(1+Az)/(1+Bz)$. In other words $f\in P^*(A,B)$ if and only if there exists a function ω regular in U and satisfying $\omega(0)=0$ and $|\omega(z)|<1$ for $z\in U$, such that

$$f'(z)=\frac{1+A\omega(z)}{1+B\omega(z)},\quad z\in U. \tag{1.1}$$

The condition (1.1) is equivalent to

$$\left|\frac{f'(z)-1}{A-Bf'(z)}\right|<1,\quad z\in U.$$

It is clear from (1.1) that $\mathrm{Re}\{f'(z)\}>0, z\in U$, and hence [4, p. 6] the members of $P^*(A,B)$ are univalent in U. Let T denote the subclass of S consisting of functions univalent in U and having Taylor expansion of the form $f(z)=z-\sum_{n=k}^{\infty}|a_n|z^n$, $k\geqslant 2$. Let us define

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$$P_k(A,B)=P^*(A,B)\cap T.$$

Let us denote by $S^*(A,B)$ and $K^*(A,B)$, the subclasses of S obtained by replacing $f'(z)$ in (1.1) by $zf'(z)/f(z)$ and $(1+zf''(z)/f'(z))$ respectively. Let

$$S_k(A,B)=S^*(A,B)\cap T \text{ and } K_k(A,B)=K^*(A,B)\cap T.$$

Author and Shukla [5] have recently studied the classes $S_k(A,B)$ and $K_k(A,B)$ when $-1\leqslant B\leqslant 0$. In this paper, under the assumption $-1\leqslant B\leqslant 0$, we obtain coefficient estimate, distortion and covering theorems and radius of convexity for the class $P_k(A,B)$. While determining the radius of convexity for $P_k(A,B)$, we have been able to extend some known results of Hallenbeck [2] and MacGregor [3]. We also obtain the class preserving integral operators of the form

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt,\quad c>-1 \tag{1.2}$$

for $P_k(A,B)$. Conversely, when $F\in P_k(A,B)$, we determine the radius of univalence of f defined by (1.2). Lastly, we show that the class $P_k(A,B)$ is closed under 'arithmetic mean' and 'convex linear combinations'.

Note Throughout this paper we assume that $-1\leqslant B\leqslant 0$ and $k\geqslant 2$.

2. Coefficient estimate

Theorem 1 A function $f(z)=z-\sum_{n=k}^{\infty}|a_n|z^n$ is in $P_k(A,B)$ if and only if

$$\sum_{n=k}^{\infty}n(1-B)|a_n|\leqslant(A-B).$$

This result is sharp.

Proof Let $|z|=1$. Then

$$\begin{aligned}|f'(z)-1|-|A-Bf'(z)| &= \Big|-\sum_{n=k}^{\infty}n|a_n|z^{n-1}\Big|-\Big|(A-B)+B\sum_{n=k}^{\infty}n|a_n|z^{n-1}\Big|\\ &\leqslant \sum_{n=k}^{\infty}n(1-B)|a_n|-(A-B),\ \text{since } -1\leqslant B\leqslant 0\\ &\leqslant 0,\ \text{by assumption.}\end{aligned}$$

Hence, by maximum modulus principle, $f\in P_k(A,B)$.

To prove the converse, let

$$\left|\frac{f'(z)-1}{A-Bf'(z)}\right|=\left|\frac{-\sum_{n=k}^{\infty}n|a_n|z^{n-1}}{(A-B)+B\sum_{n=k}^{\infty}n|a_n|z^{n-1}}\right|<1,\qquad z\in U.$$

Since $|\mathrm{Re}(z)|\leqslant|z|$ for all z, we have

$$\mathrm{Re}\left\{\frac{\sum_{n=k}^{\infty}n|a_n|z^{n-1}}{(A-B)+B\sum_{n=k}^{\infty}n|a_n|z^{n-1}}\right\}<1. \tag{2.1}$$

Choose values of z on the real axis so that $f'(z)$ is real. Upon clearing the denominator in (2.1) and letting $z\to 1$ through real values, we obtain

$$\sum_{n=k}^{\infty} n|a_n| \leqslant (A-B)+B\sum_{n=k}^{\infty} n|a_n|.$$

This completes the proof of theorem.

Sharpness follows if we take

$$f(z)=z-\frac{A-B}{n(1-B)}z^n,\quad n\geqslant k.$$

3. Distortion properties

Theorem 2 If $f\in P_k(A,B)$, then for $|z|=r$

$$r-\frac{A-B}{k(1-B)}r^k\leqslant |f(z)|\leqslant r+\frac{A-B}{k(1-B)}r^k \tag{3.1}$$

and

$$1-\frac{A-B}{1-B}r^{k-1}\leqslant |f'(z)|\leqslant 1+\frac{A-B}{1-B}r^{k-1}. \tag{3.2}$$

All these inequalities are sharp.

Proof Let $f(z)=z-\sum_{n=k}^{\infty}|a_n|z^n$. Then, it follows from Theorem 1 that

$$\sum_{n=k}^{\infty}|a_n|\leqslant\frac{A-B}{k(1-B)}.$$

Hence

$$|f(z)|\leqslant r+\sum_{n=k}^{\infty}|a_n|r^n\leqslant r+r^k\sum_{n=k}^{\infty}|a_n|\leqslant r+\frac{A-B}{k(1-B)}r^k$$

and

$$|f(z)|\geqslant r-\sum_{n=k}^{\infty}|a_n|r^n\geqslant r-r^k\sum_{n=k}^{\infty}|a_n|\geqslant r-\frac{A-B}{k(1-B)}r^k.$$

Thus (3.1) follows. Further

$$|f'(z)|\leqslant 1+\sum_{n=k}^{\infty}n|a_n|r^{n-1}\leqslant 1+r^{k-1}\sum_{n=k}^{8}n|a_n| \tag{3.3}$$

and

$$|f'(z)|\geqslant 1-\sum_{n=k}^{\infty}n|a_n|r^{k-1}\geqslant 1-r^{k-1}\sum_{n=k}^{\infty}n|a_n|. \tag{3.4}$$

But, from Theorem 1, it holds that

$$\sum_{n=k}^{\infty}n|a_n|\leqslant\frac{A-B}{1-B}. \tag{3.5}$$

The inequalities in (3.2) follow now by using (3.5) in (3.3) and (3.4).

Equality in (3.1) and (3.2) is obtained if we take

$$f(z)=z-\frac{A-B}{k(1-B)}z^k. \tag{3.6}$$

Note For the above function, equality on the left hand side of (3.1) is obtained at $z=r$ whereas on the right hand side equality is obtained at $z=-r$ when $k=2,4,6,\cdots$; $z=ir$ when $k=3,7,11,\cdots$ and $z=re^{i\pi/(k-1)}$ when $k=5,9,13,\cdots$. Similarly, the points where equality holds in (3.2) can be obtained.

Corollary 1 If $f\in P_k(A,B)$, then the disc U is mapped by f onto a domain that contains the disc $|w|<\frac{k(1-B)-(A-B)}{k(1-B)}$. The result is sharp with extremal function f given by (3.6).

The above corollary follows if we let $r\to 1$ in the left hand side inequality in (3.1). An interesting case appears when $A=1$. In this case $|w|<(k-1)/k$.

4. Integral operators

Theorem 3 Let c be a real number such that $c>-1$. If $f\in P_k(A,B)$, then the function F defined by

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt \tag{4.1}$$

also belongs to $P_k(A,B)$.

Proof Let $f(z)=z-\sum_{n=k}^{\infty}|a_n|z^n$. Then from the representation of F, it follows that

$$F(z)=z-\sum_{n=k}^{\infty}|b_n|z^z,$$

where

$$|b_n|=\left(\frac{c+1}{c+n}\right)|a_n|.$$

Therefore

$$\sum_{n=k}^{\infty}n(1-B)|b_n|=\sum_{n=k}^{\infty}n(1-B)\left(\frac{c+1}{c+n}\right)|a_n|\leqslant\sum_{n=k}^{\infty}n(1-B)|a_n|\leqslant(A-B),$$

since $f\in P_k(A,B)$. Hence, by Theorem 1, $F\in P_k(A,B)$.

Theorem 4 Let c be a real number suah that $c>-1$. If $F\in P_k(A,B)$, then the function f defined by (4.1) is univalent in $|z|<R^*$, where

$$R^*=\inf_{n>k}\left[\left(\frac{c+1}{c+n}\right)\left(\frac{1-B}{A-B}\right)\right]^{1/(n-1)}.$$

This result is sharp.

Proof Let $F(z)=z-\sum_{n=k}^{\infty}|a_n|z^n$. It followa then from (4.1) that

$$f(z)=z^{1-c}[z^cF(z)]'/(c+1)=z-\sum_{n=k}^{\infty}\left(\frac{c+n}{c+1}\right)|a_n|z^n.$$

In order to obtain the required result it suffices to show that $|f'(z)-1|<1$ in $|z|<R^*$.

Now

$$|f'(z)-1|=\left|-\sum_{n=k}^{\infty}n\left(\frac{c+n}{1+c}\right)|a_n|z^{n-1}\right|\leqslant\sum_{n=k}^{\infty}n\left(\frac{c+n}{c+1}\right)|a_n||z|^{n-1}.$$

Thus

$$|f'(z)-1|<1,\ \text{if}$$

(4.2)
$$\sum_{n=k}^{\infty}n\left(\frac{c+n}{c+1}\right)|a_n||z|^{n-1}<1.$$

But Theorem 1 confirms that

$$\sum_{n=k}^{\infty}n\left(\frac{1-B}{A-B}\right)|a_n|\leqslant 1.$$

Hence (4.2) will be satisfied if

$$n\left(\frac{c+n}{c+1}\right)|a_n||z|^{n-1}<n\left(\frac{1-B}{A-B}\right)|a_n|,\quad n=k,\ k+1,\cdots$$

or if

$$|z|<\left[\left(\frac{c+1}{c+n}\right)\left(\frac{1-B}{A-B}\right)\right]^{1/(n-1)},\ n=k,\ k+1,\ \cdots$$

Therefore f is univalent in $|z|<R^*$. Sharpness follows if we take

$$F(z)=z-\frac{A-B}{n(1-B)}z^n,\ n\geqslant k.$$

5. Radius of convexity

Theorem 5 If $f\in P_k(A,B)$, then f is convex in the disc $|z|<R$, where

$$R=\inf_{n\geqslant k}\left[\frac{1-B}{n(A-B)}\right]^{1/(n-1)}.$$

The result is sharp.

Proof In order to establish the required result it suffices to show that $|zf''(z)/f'(z)|<1$ in $|z|<R$. Let $f(z)=z-\sum_{n=k}^{\infty}|a_n|z^n$. Then we have

$$\left|\frac{zf''(z)}{f'(z)}\right|=\left|\frac{-\sum_{n=k}^{\infty}n(n-1)|a_n|z^{n-1}}{1-\sum_{n=k}^{\infty}n|a_n|z^{n-1}}\right|\leqslant\frac{\sum_{n=k}^{\infty}n(n-1)|a_n||z|^{n-1}}{1-\sum_{n=k}^{\infty}n|a_n||z|^{n-1}}.$$

Therefore $|zf''(z)/f'(z)|<1$, if

$$\sum_{n=k}^{\infty}n(n-1)|a_n||z|^{n-1}<1-\sum_{n=k}^{\infty}n|a_n||z|^{n-1}$$

or if

(5.1)
$$\sum_{n=k}^{\infty}n^2|a_n||z|^{n-1}<1.$$

Also, by Theorem 1, we have

$$\sum_{n=k}^{\infty} n\left(\frac{1-B}{A-B}\right)|a_n| \leqslant 1.$$

Hence (5.1) will be satisfied if

$$n^2|a_n||z|^{n-1} < n\left(\frac{1-B}{A-B}\right)|a_n|, \quad n=k, \ k+1, \cdots$$

or if

$$|z| < \left[\frac{1-B}{n(A-B)}\right]^{1/(n-1)}, \quad n=k, \ k+1, \cdots.$$

Therefore f is convex in $|z|<R$. Sharpness follows if we take

$$f(z) = z - \frac{A-B}{n(1-B)} z^n, \ n \geqslant k.$$

Since $P_2(1,-1)$ and $P_2(0,-1)$ are the classes of functions $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$ that satisfy the conditions $\mathrm{Re}\{f'(z)\}>0$ and $\mathrm{Re}\{f'(z)\}>\frac{1}{2}$ respectively, we have the following corollaries as the direct consequences of Theorem 5.

Corollary 2 Let $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$. If $\mathrm{Re}\{f'(z)\}>0$, then f is convex in $|z|<\frac{1}{2}$. The result is sharp with the extremal function $f(z)=z-z^2/2$.

Remark The above corollary extends a result of MacGregor [3] for the functions having negative coefficients. In fact, MacGregor proved that, if $f \in S$ and $\mathrm{Re}\{f'(z)\}>0$, then f is convex in $|z|<\sqrt{2}-1$ (=.414 approximate).

Corollary 3 Let $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$. If $\mathrm{Re}\{f'(z)\}>\frac{1}{2}$, then f is convex in $|z|<2^{-1/3}$. The result is sharp with the extremal function $f(z)=z-z^4/8$.

Remark The above corollary extends a result of Hallenbeck [2] for the functions having negative coefficients. In fact, Hallenbeck proved that, if $f \in S$ and $\mathrm{Re}\{f'(z)\}>\frac{1}{2}$, then f is convex in $|z|<2^{-\frac{1}{2}}$.

6. Closure properties

In this section we show that the class $P_k(A,B)$ is closed under 'arithmetic mean' and 'convex linear combinations'.

Theorem 6 Let $f_j(z)=z-\sum_{n=k}^{\infty}|a_{nj}|z^n$, $j=1,2,\cdots,m$, If $f_j \in P_k(A,B)$ for each $j=1,2,\cdots,m$, then the function $h(z)=z-\sum_{n=k}^{\infty}|b_n|z^n$ also belongs to $P_k(A,B)$, where $b_n=\frac{1}{m}\sum_{j=1}^{m}a_{nj}$.

Proof Since $f_j \in P_k(A,B)$, it follows from Theorem 1 that

$$\sum_{n=k}^{\infty} n(1-B)|a_{n_j}| \leqslant (A-B), \quad j=1,2,3,\cdots,m.$$

Therefore

$$\sum_{n=k}^{\infty} n(1-B)|b_n| \leqslant \sum_{n=k}^{\infty}\left[n(1-B)\left\{\frac{1}{m}\sum_{j=1}^{m} |a_{n_j}| \right\}\right] \leqslant (A-B).$$

Hence, by Theorem 1, $h \in P_k(A,B)$.

Theorem 7 Let

$$f_1(z)=z, \quad f_n(z)=z-\frac{A-B}{n(1-B)} \quad (n=k,k+1,\cdots).$$

Then $f \in P_k(A,B)$ if and only if it can be expressed in the form

$$f(z)=\lambda_1 f_1(z)+\sum_{n=k}^{\infty}\lambda_n f_n(z),$$

where $\lambda_n \geqslant 0$ and $\lambda_1+\sum_{n=k}^{\infty}\lambda_n=1$.

Proof Let us suppose that

$$f(z)=\lambda_1 f_1(z)+\sum_{n=k}^{\infty}\lambda_n f_n(z)=z-\sum_{n=k}^{\infty}\frac{A-B}{n(1-B)}\lambda_n z^n.$$

Then

$$\sum_{n=k}^{\infty}\left[\{n(1-B)\}\left\{\frac{A-B}{n(1-B)}\right\}\lambda_n\right]=(A-B)\sum_{n=k}^{\infty}\lambda_n \leqslant (A-B).$$

Hence, by Theorem 1, $f \in P_k(A,B)$.

Conversely, let $f \in P_k(A,B)$. It follows then from Theorem 1 that

$$|a_n| \leqslant \frac{A-B}{n(1-B)}, \quad (n=k,k+1,\cdots).$$

etting

$$\lambda_n=\frac{n(1-B)}{A-B}|a_n|, \quad (n=k,k+1,\cdots).$$

and

$$\lambda_1=1-\sum_{n=k}^{\infty}\lambda_n,$$

we have

$$f(z)=\lambda_1 f_1(z)+\sum_{n=k}^{\infty}\lambda_n f_n(z).$$

This completes the proof of theorem.

Our results generalize the results of Gupta and Jain [1].

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