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A Remark on the Singularities of Solutions for Three-dimensional Nonlinear Wave Equations*

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Synopsis

As a remark on [1], it is pointed out that C^2 solutions of $u_{tt}-\sum_1^3 u_{x_ix_i}=u_t^2$ with compact supported initial data must vanish everywhere in $\mathbf{R}_+^4$.

A remark on Fritz John's paper [1] is given in this paper.

In [2], a remarkable result is obtained that the Cauchy problem for nonlinear wave equation with small initial data always has global solution if the space dimension$\geqslant 6$. For the one-dimensional wave equation, the situation is just the opposite[3]. In[1], it is pointed out that for three-dimensional wave equation we can also get the negative conclusion. As an example, it is proved in[1] that C^3 solution of equation

$$u_{tt}-\sum_i^3 u_{x_ix_i}=u_t^2 \tag{1}$$

with compact supported $u(x,0)$, $u_t(x,0)$ must vanish in $\mathbf{R}_+^4$. That is the Cauchy problem

$$\begin{cases}(1)\\ t=0,\ \ u=\varphi,\ \ u_t=\psi\end{cases} \tag{2}$$

with compact supported φ, $\psi\not\equiv 0$ can not have C^3 global solution. Besides, the upper bound and the lower bound of the span life for the symmetric C^2 solution of (1) with small initial data are estimated in [1]. Therefore C^2 symmetric global solution for (1) with small initial data can not exist as well. From [1], however, we don't know whether (1) can possess C^2 solution with compact supported initial data. And the answer can be easily gained by a similar method as in [1]. But it's much simpler. Our remark gives the conclusion (See the theorem given later). This conclusion consummates the result for (1), and on the other hand it simplifies the proof of three-dimensional counter example of the result in [2].

Theorem $u\in C^2(\mathbf{R}_+^4)$ satisfying (1) with compact supported $u(x,0)$, $u_t(x,0)$ must vanish everywhere in $\mathbf{R}_+^4$.

Proof Suppose that $\mathrm{supp}(u|_{t=0})$, $\mathrm{supp}(u_t|_{t=0})$ are contained in a ball S_R with radius R. Then from theorem 4 (uniqueness theorem) of [1],

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$$u(x,t)\equiv 0,\ \text{for}\ \sqrt{\sum_1^3 x_i^2}\geqslant t+R. \tag{3}$$

For a continuous function $U(x,t)$, let

$$\overline{U}(r,t)=\frac{1}{4\pi r^2}\iint_{\Sigma\xi_i^2=r^2} U(\xi,t)\,ds. \tag{4}$$

It's an even function of r.

From (1), we have

$$(r\overline{u})_{tt}-(r\overline{u})_{rr}=r\overline{u_t^2}, \tag{5}$$

for $r\in\mathbf{R}^1$, $t>0$, and for $r\geqslant 0$

$$(r\overline{u})_{tt}-(r\overline{u})_{rr}\geqslant r\overline{u}_t^2. \tag{6}$$

Let $(r,t)\in\sum=\{(r,t)\,|\,3r-t\geqslant R,\ t-r\geqslant R\}$. Let $T_{r,t}$ denote the triangle with vertices (r,t), $(r-t,0)$, $(r+t,0)$. Then

$$\overline{u}(r,t)=\iint_{T_{r,t}}\frac{\rho}{2r}\overline{u_\tau^2}\,d\rho d\tau=\iint_{T_{r,t}\backslash T_{0,t-r}}\frac{\rho}{2r}\overline{u_\tau^2}\,d\rho d\tau$$

$$\geqslant\iint_{T_{r,t}\backslash T_{0,t-r}}\frac{\rho}{2r}\overline{u}_\tau^2\,d\rho d\tau\geqslant\iint_{\substack{\frac{t-r+R}{2}\leq\rho\leq r\\ \rho-R\leq\tau\leq\rho+t-r}}\frac{\rho}{2r}\overline{u}_\tau^2\,d\rho d\tau$$

$$\geqslant\int_{\frac{t-r+R}{2}}^{r}\frac{\rho}{2r(R+t-r)}\left(\int_{\rho-R}^{\rho+t-r}\overline{u}_\tau(\rho,\tau)\,d\tau\right)^2 d\rho=\frac{\rho}{2r(t+R-r)}\int_{\frac{t-r+R}{2}}^{r}\rho\overline{u}^2(\rho,\rho+t-r)\,d\rho. \tag{7}$$

On straight lines

$$t-r=c\in(R,\infty),$$

(7) becomes

$$\overline{u}(r,r+c)\geqslant\frac{1}{2(R+c)r}\int_{\frac{R+C}{2}}^{r}\rho\overline{u}^2(\rho,\rho+c)\,d\rho. \tag{8}$$

Let $\beta(r)=\int_{\frac{R+C}{2}}^{r}\rho\overline{u}^2(\rho,\rho+c)\,d\rho$, we have

$$\beta'(r)\geqslant\frac{1}{4(R+c)^2 r}\beta^2(r). \tag{9}$$

From this inequality we get $\beta(r)\equiv 0$. Hence $\overline{u}(r,r+c)\equiv 0$ for $c\geqslant R$, $r\geqslant R$ and $\overline{u}(r,t)\equiv 0$ in $\sum$.

From (7), we have

$$u_t(x,t)\equiv 0$$

for $\sqrt{\sum_1^3 x_i^2}+t\geqslant R$. Therefore $u_t\equiv u_{tt}\equiv 0$ for $t>R$. From (1) we have

$$\Delta u=0,\ \text{for}\ t>R. \tag{10}$$

Because of (3), $u\equiv 0$ for $t\geqslant\mathbf{R}$.

Again from theorem 4 of [1], $u\equiv 0$.

The proof is completed.

References

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