Note on a Class of Inverse Series Relations

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Let $\Gamma \equiv (\Gamma, +, \cdot)$ be the commutative ring of formal power series with real or complex coefficients, in which the ordinary addition and Cauchy multiplication are defined. For $\phi, \psi \in \Gamma$ the composition $\phi(\psi(t))$ just means a formal substitution of $u = \psi(t)$ into $\phi(u)$ in which the operations $+$ and $\cdot$ can be performed indefinitely. We have the following

Theorem Let $\phi(\psi(t)) = t$ with $\phi(0) = \psi(0) = 0$, and let

$$\frac{1}{k!}(\phi(t))^k = \sum_{n=0}^{\infty} \frac{t^n}{n!} A_1(n,k), \quad \frac{1}{k!}(\psi(t))^k = \sum_{n=0}^{\infty} \frac{t^n}{n!} A_2(n,k).$$

Then we have the pair of inverse relations

$$a_n = \sum_{k=0}^{n} A_1(n,k) b_k, \qquad b_n = \sum_{k=0}^{n} A_2(n,k) a_k.$$

This is an extension of a previous result (cf. Theorem 4 and Corollary 2 of [1]). Some related results will appear elsewhere, including a detailed investigation on the following

Problem How to obtain a kind of reciprocal integral transforms analogous to the inverse relations just mentioned in the theorem?

Reference

[1] Hsu, L. C., Self-reciprocal functions and self-reciprocal transforms, This Journal, №2 (1981), 119—138.