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The Number of Nontrivial Solutions of Nonlinear Two Point Boundary Value Problems*

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In this paper we use the Leray-Schauder degree theory to investigate the number of nontrivial solutions of the nonlinear two point boundary value problem

$$\begin{cases} \dfrac{d^2x}{dt^2}+f(x)=0, \quad 0\leqslant t\leqslant 1; \\ x(0)=x(1)=0, \end{cases} \tag{1}$$

where $f(x)$ is non-negative and continuous for $0\leqslant x<+\infty$ and $f(0)=0$. Obviously, $x(t)\equiv 0$ is a (trivial) solution of (1).

Theorem 1 If

$$0\leqslant \overline{\lim_{x\to+0}}\frac{f(x)}{x}<8 \tag{2}$$

and

$$24\sqrt{3}<\underline{\lim_{x\to+\infty}}\frac{f(x)}{x}\leqslant+\infty, \tag{3}$$

then the problem (1) has at least one nontrivial solution $x(t)\in C^2[0,\ 1]$ satisfying $x(t)>0\ (\forall\, 0<t<1)$.

Proof It is well known that the solution (in $C^2[0,\ 1]$) of problem (1) is equivalent to the solution (in $C[0,\ 1]$) of the Hammerstein integral equation

$$x(t)=\int_0^1 G(t,\ s)f(x(s))ds=Ax(t), \tag{4}$$

where $G(t,\ s)$ denotes the corresponding Green function:

$$G(t,s)=\begin{cases} t(1-s), & t\leqslant s; \\ s(1-t), & t>s. \end{cases}$$

Let $P=\{x(t)\mid x(t)\in C[0,\ 1],\ x(t)\geqslant 0\}$ and $P_\varepsilon=\{x(t)\mid x(t)\in P,$

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$\min\limits_{1/2-\varepsilon\leqslant t\leqslant 1/2+\varepsilon} x(t)\geqslant(1/2-\varepsilon)\,\|x\|_c\}$ for $0<\varepsilon<1/2$. It it easy to see that P and P_ε are cones in $C[0,1]$ ($P_\varepsilon\subset P$) and A is completely continuous from P into P.

Suppose that $x(t)\in P$. Observing $G(t,s)\leqslant s(1-s)$ we obtain

$$\|Ax\|_c\leqslant\int_0^1 s(1-s)f(x(s))ds. \tag{5}$$

On the other hand, for $1/2-\varepsilon\leqslant t\leqslant 1/2+\varepsilon$, we have

$$G(t,s)=\begin{cases} t(1-s)\geqslant(1/2-\varepsilon)(1-s), & t\leqslant s;\\ s(1-t)\geqslant s[1-(1/2+\varepsilon)]=(1/2-\varepsilon)s, & t>s,\end{cases}$$

and therefore

$$G(t,s)\geqslant\left(\frac{1}{2}-\varepsilon\right)s(1-s),\qquad \forall\,\frac{1}{2}-\varepsilon\leqslant t\leqslant\frac{1}{2}+\varepsilon,\ 0\leqslant s\leqslant 1,$$

hence

$$\min_{\frac{1}{2}-\varepsilon\leqslant t\leqslant\frac{1}{2}+\varepsilon} Ax(t)\geqslant\left(\frac{1}{2}-\varepsilon\right)\int_0^1 s(1-s)f(x(s))ds. \tag{6}$$

It follows from (5) and (6) that

$$\min_{\frac{1}{2}-\varepsilon\leqslant t\leqslant\frac{1}{2}+\varepsilon} Ax(t)\geqslant\left(\frac{1}{2}-\varepsilon\right)\|Ax\|_c\,,$$

i. e., $Ax(t)\in P_\varepsilon$. Thus, $A(P)\subset P_\varepsilon$, and hence

$$A(P_\varepsilon)\subset P_\varepsilon,\quad \forall\,0<\varepsilon<\frac{1}{2}. \tag{7}$$

By virtue of (2) and $f(0)=0$, there exist $r>0$ and $0<\tau<8$ such that

$$0\leqslant f(x)\leqslant(8-\tau)x,\quad \forall\,0\leqslant x\leqslant r. \tag{8}$$

Now we prove that

$$Ax(t)\not\geqslant x(t),\quad \forall\,x(t)\in P,\ \|x\|_c=r. \tag{9}$$

In fact, if (9) is not true, there exists $x_0(t)\in P$, $\|x_0\|_c=r$ such that $Ax_0(t)\geqslant x_0(t)$, then

$$x_0(t)\leqslant Ax_0(t)\leqslant(8-\tau)\int_0^1 G(t,s)x_0(s)ds\leqslant(8-\tau)\|x_0\|_c\int_0^1 G(t,s)ds$$
$$=\left(4-\frac{\tau}{2}\right)t(1-t)\|x_0\|_c\leqslant\left(1-\frac{\tau}{8}\right)\|x_0\|_c\,,$$

hence $\|x_0\|_c\leqslant\left(1-\frac{\tau}{8}\right)\|x_0\|_c<\|x_0\|_c$, which is a contradiction, and therefore (9) holds.

By virtue of (3) there exist $\eta>0$ and $\sigma>0$ such that

$$f(x)\geqslant(24\sqrt{3}+\sigma)x,\quad \forall\,x>\eta. \tag{10}$$

Choose

$$R_\varepsilon > \max\left\{r, \eta\left(\frac{1}{2} - \varepsilon\right)^{-1}\right\}. \tag{11}$$

we prove that

$$Ax(t) \not\leqslant 12\sqrt{3}\,\varepsilon(1-\varepsilon)\left(\frac{1}{2} - \varepsilon\right)x(t), \quad \forall x(t) \in P_\varepsilon, \ \|x\|_c = R_\varepsilon. \tag{12}$$

Suppose that (12) is not true. Then, there exists $x^*(t) \in P_\varepsilon$, $\|x^*\|_c = R_\varepsilon$ such that

$$Ax^*(t) \leqslant 12\sqrt{3}\,\varepsilon(1-\varepsilon)\left(\frac{1}{2} - \varepsilon\right)x^*(t). \tag{13}$$

Observing

$$\min_{\frac{1}{2}-\varepsilon \leqslant t \leqslant \frac{1}{2}+\varepsilon} x^*(t) \geqslant \left(\frac{1}{2} - \varepsilon\right)\|x^*\|_c = \left(\frac{1}{2} - \varepsilon\right)R_\varepsilon > \eta,$$

we have

$$\begin{aligned} Ax^*\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right) f(x^*(s))\,ds \geqslant \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} G\left(\frac{1}{2}, s\right) f(x^*(s))\,ds \\ &\geqslant (24\sqrt{3} + \sigma)\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} G\left(\frac{1}{2}, s\right) x^*(s)\,ds \geqslant (24\sqrt{3} + \sigma) \\ &\quad \times \left(\frac{1}{2} - \varepsilon\right)\|x^*\|_c \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} G\left(\frac{1}{2}, s\right) ds \\ &= (12\sqrt{3} + \sigma/2)(1/2 - \varepsilon)\,\varepsilon(1-\varepsilon)\|x^*\|_c. \end{aligned} \tag{14}$$

From (13) and (14) we obtain

$$\left(12\sqrt{3} + \frac{\sigma}{2}\right)\|x^*\|_c \leqslant 12\sqrt{3}\,x^*\left(\frac{1}{2}\right) \leqslant 12\sqrt{3}\,\|x^*\|_c,$$

which is a contradiction. Hence, (12) holds.

It is easy to know that the fnnction $\varphi(\varepsilon) = \varepsilon(1-\varepsilon)(1/2 - \varepsilon)$ attains its maximum in $0 < \varepsilon < 1/2$ at $\varepsilon = \varepsilon_0 = \frac{3-\sqrt{3}}{6}$ and $\varphi(\varepsilon_0) = \frac{\sqrt{3}}{36}$. Put $\varepsilon = \varepsilon_0$ in (12), we obtain

$$Ax(t) \not\leqslant x(t), \quad \forall x(t) \in P_{\varepsilon_0}, \ \|x\|_c = R_{\varepsilon_0}. \tag{15}$$

Observing (7), (9) and (15) and using the fixed point theorem of cone expansion (see [2] Theorem 45.1 or [3] Corollary 12.5), we assert that A has a fixed point $x(t) \in P_{\varepsilon_0}$ satisfying $r < \|x\|_c < R_{\varepsilon_0}$ and our theorem is proved.

Remark 1 Theorem 1 can not be deduced from the results of [1], since the Green function $G(t, s)$ does not satisfy the conditions in [1].

Remark 2 It is easy to point out some elementary functions $f(x)$, which satisfy the conditions of theorem 1; for example,

$$f(x)=\sum_{i=1}^{n}a_ix^i,\ a_i\geqslant 0(i=1,\ 2,\ \cdots,\ n),\ a_1<8,\ a_n>0,\ n>1,$$

$$f(x)=\frac{42x^2(2-\cos x)}{1+x}.$$

Theorem 2 Let (3) be satisfied and

$$24\sqrt{3}<\varliminf_{x\to+0}\frac{f(x)}{x}\leqslant+\infty. \tag{16}$$

Suppose that there exist R>O such that

$$\max_{0\leqslant x\leqslant R}f(x)<8R. \tag{17}$$

Then the problem (1) has at least two nontrivial solutions $x_1(t)\in C^2[0,\ 1]$ and $x_2(t)\in C^2[0,\ 1]$ satisfying $x_1(t)>0$ and $x_2(t)>0(\forall\, 0<t<1)$.

Proof we use the notations in the proof of theorem 1. It is easy to see that (7) also holds now. Observing (17) and using the method similar to the proof of (9), we can deduce that

$$Ax(t)\not\geqslant x(t),\ \ \forall x(t)\in P,\ \|x\|_c=R. \tag{18}$$

On the other hand, observing (3) and (16) and using the method similar to the proof of (15), we can assert that there exist $R_0>R>r_0>0$ such that

$$Ax(t)\not\leqslant x(t),\ \ \forall x(t)\in P_{\varepsilon_0},\ \|x\|_c=R_0 \tag{19}$$

and

$$Ax(t)\not\leqslant x(t),\ \ \forall x(t)\in P_{\varepsilon_0},\ \|x\|_c=r_0. \tag{20}$$

Now, by the fixed point theorem of cone expansion anu compression it follows from (19), (18) and (20) that there exist $x_1(t)\in P_{\varepsilon_0}$ and $x_2(t)\in P_{\varepsilon}$ such that $Ax_i(t)=x_i(t)\ (i=1,\ 2)$ and $R_0>\|x_1\|_c>R>\|x_2\|_c>r_0$. Our theorem is proved.

Remark 3 It is easy to point out some elementary functions $f(x)$, which satisfy the conditions of theorem 2; for example,

$$f(x)=x^\alpha+x^\beta\qquad(\beta>1>\alpha>0),$$
$$f(x)=e^x\ln(1+\sqrt{x}).$$

In both cases we may choose $R=1$.

In the following theorem we assume that $f(x)$ is defined and continuous in $-\infty<x<+\infty$ and $f(0)=0$.

Theorem 3 If

$$xf(x)\geqslant 0,\ \ \forall -\infty<x<+\infty, \tag{21}$$

$$0\leqslant\varlimsup_{x\to 0}\frac{f(x)}{x}<8 \tag{22}$$

and

$$24\sqrt{3}<\varliminf_{x\to\infty}\frac{f(x)}{x}\leqslant+\infty \tag{23}$$

then the problem (1) has at least two nontrivial solutions $x_1(t)\in C^2[0,1]$ and $x_2(t)\in C^2[0,1]$ satisfying $x_1(t)>0$ and $x_2(t)<0\ (\forall\, 0<t<1)$.

Proof From (21) we have

$$f(x)\geqslant 0,\quad \forall x>0;\quad f(x)\leqslant 0,\quad \forall x<0. \tag{24}$$

Hence, theorem 1 implies that problem (1) has a solution $x_1(t)\in C^2[0,1]$ satisfying $x_1(t)>0\ (\forall\, 0<t<1)$.

Now, let $g(x)=-f(-x)$, we find from (24), (22) and (23) that

$$g(x)\geqslant 0,\quad \forall x>0;\quad g(0)=0,$$

$$0\leqslant\varlimsup_{x\to+0}\frac{g(x)}{x}=\varlimsup_{x\to+0}\frac{f(-x)}{-x}<8$$

and

$$24\sqrt{3}<\varliminf_{x\to+\infty}\frac{g(x)}{x}=\varliminf_{x\to+\infty}\frac{f(-x)}{-x}\leqslant+\infty.$$

Hence, theorem 1 implies that the problem

$$\begin{cases}\dfrac{d^2x}{dt^2}+g(x)=0,\quad 0\leqslant t\leqslant 1;\\ x(0)=x(1)=0\end{cases}$$

has a solution $x^*(t)\in C^2[0,1]$ satisfying $x^*(t)>0\ (\forall\, 0<t<1)$.
Put $x_2(t)=-x^*(t)$. It is evident that $x_2(t)$ is a solution of problem (1) satisfying $x_2(t)<0\ (\forall\, 0<t<1)$ and our theorem is proved.

Remark 4 It is easy to point out some elementary functions $f(x)$, which satisfy the conditions of theorem 3; for example,

$$f(x)=x^3+x^5(1-\sin x),$$

$$f(x)=x^4\ln\left(1+\frac{x}{1+x^2}\right).$$

In the same way we can discuss the number of nontrivial solutions of the problem

$$\begin{cases}\dfrac{d^2x}{dt^2}+f(x)=0,\quad 0\leqslant t\leqslant 1;\\ x(0)=x'(1)=0\end{cases}$$

and establish three similar theorems. At this time, the corresponding Green function is

$$G(t,s)=\begin{cases} t, & t\leqslant s; \\ s, & t>s \end{cases}$$

and the corresponding cones are $P=\{x(t)\,|\,x(t)\in C[0,1],\ x(t)\geqslant 0\}$ and $P_\varepsilon=\{x(t)\,|\,x(t)\in P,\ \min\limits_{\varepsilon<t<1} x(t)\geqslant \varepsilon\|x\|_C\}$ $(0<\varepsilon<1)$.

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