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On a Probabilistic Representation Theorem of Operator Semigroups with Bounded Generator*

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In this paper we derive an extension of a general probabilistic representation formula for strongly continuous operator semigroups $\{T(t); t\geqslant 0\}$ presented in [3], valid for semigroups with bounded infinitesimal generator A. Semigroups of this kind play an important role in probability theory, especially in the field of homogeneous Markov jump processes (see for example [4], [2]). For simplicity, our notation will closely follow [3]. (Note that in [3], the semigroup $\{T(t); t\geqslant 0\}$ was erroneously identified with a Banach algebra $\mathscr{T}$. Clearly, $\mathscr{T}$ should denote the Banach algebra generated by the semigroup.

THEOREM Let N be a non-negative inter-valued random variable with unit mean such that its characteristic function is analytic in some neighbourhood of the origin. Then if ψ_N denotes the moment generating function of N, we have for all $\xi>0$

$$T(\xi)=\lim_{n\to\infty}\left\{\psi_N\left(I+\frac{\xi}{n}A\right)\right\}^n \tag{1}$$

where I denotes the identity operator.

PROOF Heuristically, relation (1) is obtained from the following relation proved in [3]:

$$T(\xi)=\lim_{n\to\infty}\left\{\psi_N\left(T\left(\frac{\xi}{n}\right)\right)\right\}^n, \tag{2}$$

replacing $T\left(\frac{\xi}{n}\right)$ by the first two terms of the corresponding Taylor expansion ([1], Proposition 1. 1. 6).

For the proof, let $N_1, N_2, \cdots$ be independent copies of N and let $S_n=\sum_{k=1}^{n}N_k$, $n\in\mathbb{N}$. Then by [3],

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$$\left\{\psi_N\left(T\left(\frac{\xi}{n}\right)\right)\right\}^n = E\left(T\left(\frac{\xi}{n}S_n\right)\right) = E\left(T\left(\frac{\xi}{n}\right)^{S_n}\right), \tag{3}$$

and similarly,

$$\left\{\psi_N\left(I + \frac{\xi}{n}A\right)\right\}^n = \left\{E\left(I + \frac{\xi}{n}A\right)^N\right\}^n = E\left(\left(I + \frac{\xi}{n}A\right)^{S_n}\right). \tag{4}$$

Again by [1], Proposition 1. 1. 6 and the fact that in our case, $T(t) = e^{At}$, $t>0$,

$$\begin{aligned}
&\left\|\left\{\psi_N\left(T\left(\frac{\xi}{n}\right)\right)\right\}^n - \left\{\psi_N\left(I + \frac{\xi}{n}A\right)\right\}^n\right\| \leqslant E\left(\left\| T\left(\frac{\xi}{n}\right)^{S_n} - \left(I + \frac{\xi}{n}A\right)^{S_n}\right\|\right)\\
&= E\left(\left\|\sum_{k=0}^{S_n-1} T\left(\frac{\xi}{n}\right)^{S_n-k}\left(I + \frac{\xi}{n}A\right)^k - T\left(\frac{\xi}{n}\right)^{S_n-k-1}\left(I + \frac{\xi}{n}A\right)^{k+1}\right\|\right)\\
&\leqslant E\left(\sum_{k=0}^{S_n-1}\left\| T\left(\frac{\xi}{n}\right)^{S_n-k-1}\left\{T\left(\frac{\xi}{n}\right) - \left(I + \frac{\xi}{n}A\right)\right\}\left(I + \frac{\xi}{n}A\right)^k\right\|\right)\\
&\leqslant \left\| T\left(\frac{\xi}{n}\right) - \left(I + \frac{\xi}{n}A\right)\right\| E\left(e^{\frac{\xi}{n}\|A\| S_n}\frac{\left(1 + \frac{\xi}{n}\|A\|\right)^{S_n} - 1}{\frac{\xi}{n}\|A\|}\right.\\
&\leqslant \frac{n}{\xi}\|A\|\int_0^{\frac{\xi}{n}}\left(\frac{\xi}{n} - s\right)\|T(s)\| ds E\left(e^{2\frac{\xi}{n}\|A\| S_n}\right)\\
&\leqslant \frac{n}{\xi}\|A\| e^{\frac{\xi}{n}\|A\|}\int_0^{\frac{\xi}{n}}\left(\frac{\xi}{n} - s\right) ds E\left(e^{2\frac{\xi}{n}\|A\| S_n}\right)\\
&\leqslant \frac{\xi}{2n}\|A\| e^{\frac{\xi}{n}\|A\|} E\left(e^{2\frac{\xi}{n}\|A\| S_n}\right).
\end{aligned}$$

Following [3], $E(e^{2\frac{\xi}{n}\|A\| S_n}) \to e^{2\xi\|A\|}$ for $n \to \infty$, hence for sufficiently large n,

$$\left\|\left\{\psi_N\left(T\left(\frac{\xi}{n}\right)\right)\right\}^n - \left\{\psi_N\left(I + \frac{\xi}{n}A\right)\right\}^n\right\| \leqslant \frac{\xi}{n}\|A\| e^{3\xi\|A\|}. \tag{5}$$

Applying (2) now gives the desired result.

Note that by a slight modification of the foregoing proof, one could also obtain the following relation:

$$T(\xi) = \lim_{n\to\infty}\left\{\psi_N\left(I + \frac{A}{n}\right)\right\}^n, \tag{6}$$

where now N is a random variable with mean $\xi>0$.

For special choices of N, the following well-known representation formulae are reobtained:

A) $N \equiv 1$, $i.\ e.\ \psi_N(t) = 1$:

$$T(\xi) = \lim_{n\to\infty}\left(I + \frac{\xi}{n}A\right)^n.$$

B) N being geometrically distributed with unit mean, i. e. $\psi_N(t)=\frac{1}{2-t}$:

$$T(\xi)=\lim_{n\to\infty}\left(2I-\left(I+\frac{\xi}{n}A\right)\right)^{-n}=\lim_{n\to\infty}\left(I-\frac{\xi}{n}A\right)^{-n}.$$

C) N being binomially distributed with mean $\xi\in(0,1)$, i. e. $\psi_N(t)=1-\xi+\xi t$:

$$T(\xi)=\lim_{n\to\infty}\left((1-\xi)I+\xi\left(I+\frac{A}{n}\right)\right)^n=\lim_{n\to\infty}\left(I+\frac{\xi}{n}A\right)^n.$$

D) N being Poisson-distributed with mean $\xi>0$, i. e. $\psi_N(t)=e^{-\xi}e^{t\xi}$:

$$T(\xi)=\lim_{n\to\infty}\left(e^{-\xi I+\xi(I+\frac{A}{n})}\right)^n=e^{A\xi}.$$

Of course, a lot of further representation theorems can be obtained from the general formula. For instance, let N have a uniform distribution over the points $\{0,2\}$. Then $\psi_N(t)=\frac{1}{2}+\frac{1}{2}t^2$, hence

E) $$T(\xi)=\lim_{n\to\infty}\left(\frac{1}{2}I+\frac{1}{2}\left(I+\frac{\xi}{n}A\right)^2\right)^n=\lim_{n\to\infty}\left(I+\frac{\xi}{n}A+\frac{\xi^2}{2n^2}A^2\right)^n.$$

Note that in this case, $T\left(\frac{\xi}{n}\right)$ is replaced by the first three terms of the Taylor expansion in (2).

Or, if N has a uniform distribution over the points $\{0,2\xi\}$, then $\psi_N(t)=\frac{1}{2}+\frac{1}{2}t^{2\xi}$, hence

F) $$T(\xi)=\lim_{n\to\infty}\left(\frac{1}{2}I+\frac{1}{2}\left(I+\frac{A}{n}\right)^{2\xi}\right)^n \quad \text{etc.}$$

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