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On the Rational Spline Functions*

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Since Schoenberg's fundamental paper was published in 1946, the theory of polynomial spline functions developed rapidly and was rich in contents. Spline functions have wide range of applications to numerical approximation. But to certain functions, for example, to the function with singular points, the interpolation by polynomial spline is not always efficient and reasonable. Since 1973 some authors have investigated the rational spline functions, and abtained a lot of results. But if we apply these results to practice, still there will be many difficulties, since we will have to solve some more complicated nonlinear simultaneous equations in the computing process.

The purpose of this paper is to establish a general representation of rational spline and to discuss some specific forms of rational splines for practical needs. If we interpolate a function by these rational splines, we can avoid solving nonlinear equations. Some practical computation has shown that the degree of accuracy of the result obtained by our method is quite high.

§1 The General Representative of Rational Spline

Throughout the paper the collection of all polynomials of degree n is noted by H_n, and the collection of all rational functions of the form $R(x)=P(x)/Q(x)$ with $P(x)\in H_r$, $Q(x)\in H_l$, $(P(x),Q(x))=1$ is noted by R_{rl}.

Definition Let $\Delta: a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ be a partition of the interval $[a,b]$ If a real function $R(x)$ defined on $[a,b]$ satisfies the following conditions:

(i) $R(x)\in R_{rl}$ in each interval $[x_j,x_{j+1}]$;

(ii) $R(x)\in C^k[a,b]$,

then $R(x)$ is said to be a rational spline of order $(r,l)^k$ with respect to the partition Δ.

The collection of all rational splines of order $(r,l)^k$ on Δ is denoted by $R_{rl}^{(k)}(\Delta)$ or $R_{rl}^{(k)}$.

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Lemma 1.1 Suppose $x_0 \in [a,b]$ and $Q(x_0) \neq 0$ then every $R(x) \in R_{r,l}$ may be represented in the form

$$R(x)=\sum_{i=0}^{k}\frac{R^{(i)}(x_0)}{i!}(x-x_0)^i+(x-x_0)^{k+1}\frac{F(x)}{Q(x)},$$

where the $F(x)$ is a polynomial whose order is $\leqslant \max(r-k-1, l-1)$.

Using Lemma 1.1, we obtain

Theorem 1 Every rational spline function in the class $R_{r,l}^{(k)}$ may be represented in the form

$$R(x)=\frac{P_1(x)}{Q_1(x)}+\sum_{j=1}^{n-1}\frac{M_j(x)}{Q_j(x)Q_{j+1}(x)}(x-x_j)_+^{k+1}$$

where the $M_j(x) \in H_{r+l-k-1}$ can be uniquely determined by the first k-derivatives at x_j of the functions $P_1(x)$, $Q_1(x)$ and $R(x)$.

§2 Rational Spline Functions with Term $\frac{A}{x-a}$

Beginning with this section, we shall discuss some interpolating methods of rational spline. The virture of these methods is that they not only preserve peculiarity of rational splines, but also can avoid solving nonlinear equations. Now we discuss first two interpolating problems of rational spline which has the form in the subinterval $[x_j, x_{j+1}]$ of $[a, b]$ as follows:

$$R(x)=R_j(x)=P_j(x)+\frac{A_j}{x-a_j}, \tag{1}$$

where $P_j(x)$ are polynomials.

1 Suppose $P_0(x) \in H_{2k-1}$ and the interpolating Conditions are given by

$$R^{(i)}(x_j)=y_j^{(i)},\ R^{(i)}(x_{j+1})=y_{j+1}^{(i)} \quad (i=0,1,\cdots,k+1). \tag{2}$$

Now we write

$$R_j(x)=\sum_{i=0}^{k}\left[\frac{m_j^{(i)}}{(x_{j+1}-x_j)^i}p_i\left(k,\frac{x-x_j}{x_{j+1}-x_j}\right)+\frac{m_{j+1}^{(i)}}{(x_{j+1}-x_j)^i}\right.$$

$$\left.\cdot q_i\left(k,\frac{x-x_j}{x_{j+1}-x_j}\right)\right]+\frac{d_i}{x-c_i}(x-x_j)^{k+1}(x-x_{j+1})^{k+1}, \tag{3}$$

where

$$p_i(k,t)=\frac{1}{i!}\sum_{j=0}^{k-i}(-1)^{i+j-1}\binom{2k+1-i}{j}t^{i+j}(t-1)^{2k+1-i-j}, \tag{4}$$

$$q_i(k,t)=\frac{1}{i!}\sum_{j=0}^{k-i}(-1)^{j}\binom{2k+1-i}{j}t^{2k+1-i-j}(t-1)^{i+j}. \tag{5}$$

The following properties of polynomials as given by (4)、(5) have been established in [6]:

$$p_i^{(j)}(k,0)=q_i^{(j)}(k,1)=\delta_{ij},$$
$$p_i^{(j)}(k,1)=q_i^{(j)}(k,0)=0 \qquad (0\leqslant j\leqslant k)$$

and

$$q_i^{(k+1)}(k,0)=(-1)^{k+i-1}p_i^{(k+1)}(k,1)=\frac{(-1)^i(2k+1-i)!}{i!\,(k-i)!} \tag{6}$$

$$q_i^{(k+1)}(k,1)=(-1)^{k+i-1}p_i^{(k+1)}(k,0)=\frac{(-1)^i(2k+1+i)!}{i!\,(k+1-i)!}$$

It may be derived from (2) and (6) that

$$m_j^{(i)}=y_j^{(i)} \qquad (i=0,1,\cdots,k) \tag{7}$$

If the polynomial part on the right hand side of (3) are $N_j(x)$, then

$$N_j^{(k+1)}(x_j)=\sum_{i=0}^{k}\frac{1}{(x_{j+1}-x_j)^{k+1}}\Big[(-1)^{k+1}y_j^{(i)}\frac{(2k+1-i)!}{i!\,(k+1-i)!}$$
$$+(-1)^i y_{j+1}^{(i)}\frac{(2k+1-i)!}{i!\,(k-i)!}\Big],$$

$$N_j^{(k+1)}(x_{j+1})=\sum_{i=0}^{k}\frac{1}{(x_{j+1}-x_j)^{k+1}}\Big[(-1)^{k+1}y_j^{(i)}\frac{(2k+1-i)!}{i!\,(k-i)!}$$
$$+(-1)^i y_{j+1}^{(i)}\frac{(2k+1+i)!}{i!\,(k+1-i)!}\Big].$$

Then if

$$(-1)^{k+1}(y_j^{(k+1)}-N_j^{(k+1)}(x_j))+(y_{j+1}^{(k+1)}-N_{j+1}^{(k+1)}(x_j))\neq 0, \tag{8}$$
$$(y_j^{(k+1)}-N^{(k+1)}(x_j))(y_{j+1}^{(k+1)}-N_{j+1}^{(k+1)}(x_j))\neq 0,$$

by the interpolating conditions (6), we have

$$c_j=x_{j+1}-\frac{(-1)^{k+1}(x_{j+1}-x_j)[y_j^{(k+1)}-N_j^{(k+1)}(x_j)]}{[(-1)^{k+1}y_j^{k+1}-y^{(k+1)}]-[(-1)^{k+1}N_j^{(k+1)}(x_j)-N^{(k+1)}(x_{j+1})]} \tag{9}$$

$$d_j=\frac{(-1)^{k+1}[y_{j+1}^{(k+1)}-N_j^{(k+1)}(x_{j+1})][y_j^{(k+1)}-N_j^{(k+1)}(x_j)]}{(k+1)!\,(x_{j+1}-x_j)^k\{[(-1)^{k+1}y_j^{(k+1)}-y_{j+1}^{(k+1)}]-[(-1)^{k+1}N_j^{(k+1)}(x_j)-N_j^{(k+1)}(x_{j+1})]\}}.$$

Thus we have proved the following

Theorem 2 Let the condition (8) be satisfied. Then the solution of interpolation problem (1) with (2) exists and is uniquely determined.

Usually, in the practical problems, at the points $x_1, x_2, \cdots, x_{n-1}$, the value $y_j^{(0)}$ is given only, but $y_j^{(i)}(i\geqslant 1)$ are unknown. In such a case, the $y_j^{(i)}$ can be computed by the numerical differentiation.

In practice, k is generally taken to be 0, 1 and 2. For example, for $k=0$, we have

$$R_j(x)=y_j+f(x_j,x_{j+1})(x-x_j)$$
$$+\frac{(x-x_j)(x-x_{j+1})[y_j'-f(x_j,x_{j+1})][y_{j+1}'-f(x_j,x_{j+1})]}{[y_j'+y'_{j+1}-2f(x_j,x_{j+1})]x-[x_jy_j'+x_{j+1}y'_{j+1}-(x_j+x_{j+1})f(x_j,x_{j+1})]} \tag{10}$$

where $f(x_j,x_{j+1})=(y_{j+1}-y_j)/(x_{j+1}-x_j)$.

If the value y_j', y'_{j+1} are unknown, we may substitute $y'\approx[f(x_{j-1},x_j)+f(x_j,x_{j+1})]/2$ to (10), obtain

$$R_j(x)=y_j+f(x_j,x_{j+1})(x-x_j)$$
$$+\frac{(x-x_j)(x-x_{j+1})[f(x_{j-1},x_j)-f(x_j,x_{j+1})][f(x_{j+1},x_{j+2})-f(x_j,x_{j+1})]}{2[f(x_{j-1},x_j)+f(x_{j+1},x_{j+2})-2f(x_j,x_{j+1})]x+2[x_jf(x_{j-1},x_j)+x_{j+1}f(x_{j+1}x_{j+2})-(x_j+x_{j+1})f(x_j,x_{j+1})]} \tag{11}$$

It is readily verified that the formula (11) is always true for the functions 1, x and x^2, This fact show that formula (11) possesses the degree of algebraic accuracy of second order; We note that for the function $\frac{1}{x}$, the formula (11) is also true. From this fact we see that the rational spline (1) is better than the polynomial spline.

2 Suppose that the interpolating conditions are given by

$$\begin{cases}R^{(i)}(x_j)=y_j^{(i)} & (0\leqslant i\leqslant k),\\ R(x_{j+1})=y_{j+1}, \ R'(x_{j+1})=y'_{j+1};\end{cases} \tag{12}$$

or

$$\begin{cases}R^{(i)}(x_j)=y_j^{(i)} & (0\leqslant i\leqslant k+1)\\ R(x_{j+1})=y_{j+1};\end{cases} \tag{13}$$

then the existence and uniqueness theorem for the interpolating problems remains true. Now take the example of $k=0$, and

$$R_j(x)=a_j+\frac{b_j(x-x_j)}{x-c_j} \qquad (x_j\leqslant x\leqslant x_{j+1}).$$

it is readily obtained that

$$a_j=y_j,$$
$$b_j=f(x_j,x_{j+1})\left[x_{j+1}-\frac{x_jy_j'-x_{j+1}f(x_j,x_{j+1})}{y_j'-f(x_j,x_{j+1})}\right], \tag{14}$$
$$c_j=\frac{x_jy_j'-x_{j+1}f(x_j,x_{j+1})}{y_j'-f(x_j,x_{j+1})}$$

In the practical applications, the $R(x)$ will be uniquely determined if the typical values at all mesh points and the derivative of first order at point a (or b) have been given In fact, since

$$R'_j(x_{j+1})=[f(x_j,x_{j+1})]^2/y_j',$$

then with given y_0 and y_0', the functions y and $R_0(x)$ will be determined and with y_1, $R_0'(x_1)$ and y_2 $R_1(x)$ will be determined, etc.

§3 The Rational Spline with Term $\frac{A}{(cx+d)^{2k-1}}$

Now we consider the rational splines

$$R(x)=R_j(x)=\sum_{i=0}^{2k-1}a_{ij}(x-x_j)^i+\frac{(x-x_j)^k(x-x_{j+1})^k}{(c_jx+d_j)^{2k-1}} \tag{15}$$

$$(x_j\leqslant x\leqslant_{j+1},\ j=0,1,\cdots,n-1)$$

and consider two types of interpolating conditions.

1 The symmetric end conditions:

$$\left.\begin{aligned}&R^{(i)}(x_j)=y_j^{(i)},\\ &R^{(i)}(x_{j+1})=y_{j+1}^{(i)}.\end{aligned}\right\}\quad (j=0,1,\cdots,k) \tag{16}$$

we have

Theorem 3 The solution of interpolation problem (15) with (16) exists and is uniquely determined.

2 The unsymmetric end conditions.

For example, we study the case $k=1$. In such a case,

$$R_j(x)=a_j^{(0)}+a_j^{(1)}\ (x-x_j)+\frac{(x-x_j)(x-x_{j+1})}{a_j^{(2)}x+a_j^{(3)}}, \tag{17}$$

and the interpolating conditions are

$$\begin{aligned}&R(x_j)=y_j,\ R'(x_j)=y_j',\\ &R''(x_j)=y_j'',\ R(x_{j+1})=y_{j+1}.\end{aligned} \tag{18}$$

By simple calculation, we can readily obtain

$$R_j(x)=f(x_j)+f(x_j,x_{j+1})(x-x_j)+\frac{2[y'_j-f(x_j,x_{j+1})]^2(x-x_j)(x_{j+1}-x)}{2[y_j'-f(x_j,x_{j+1})](x_{j+1}-x)+y_j''(x_j-x_{j+1})(x-x_j)}, \tag{19}$$

$$(x_j\leqslant x\leqslant x_{j+1}).$$

Thus for a partition of the interval $[a,b]$, apart from the values of the functions $f(x)$ on the mesh points $x_j(j=0,1,\cdots,N)$, and assuming that $f'(a)$、$f''(a)$ (or $f'(b)$、$f''(b)$) are given, the whole curve of the rational spline $R(x)\in R_{(2,1)}^{(2)}$ will be uniquely determined. In general, the rational spline curve that is determined by (17) has the peculiarity of preserving convexity.

§4 The Rational Spline with Term $\sum_{i=1}^{l}\frac{A_{ij}}{x-a_{ij}}$

Now we discuss the rational spline of which the expression on each subinterval $[x_j,x_{j+1}]$ is given by

$$R_j(x)=G_j(x)+(x-x_j)^{k+1}\sum_{i=1}^{l}\frac{A_{ij}}{x-a_{ij}}, \tag{20}$$

and assume the interprolating conditions of the form

$$R^{(2r)}(x_j)=m_{rj} \qquad \left(0\leqslant r\leqslant\frac{k-l-1}{2}\right), \tag{21}$$

$$R^{(r)}(x_j)=m_{rj} \qquad (k-l+1\leqslant r\leqslant k),\ j=0,1,\cdots,n-1,$$

where $G_j(x)\in H_n$, the constants a_{ij} being fixed, but the A_{ij}'s being undetermined coefficient, $k+l+1$ an even number.

Lemma 4.1 If $\xi_1<\xi_2<\cdots<\xi_l$, $a_1<a_2<\cdots a_l$ and $\xi_i a_j>0$, $|\xi_i|<|a_j|$ (or $|\xi_i|>|a_j|$) for all i, j; then

$$\det\left(\frac{1}{(\xi_i-a_j)^{k+1}}\right)\neq 0.$$

Proof Suppose that

$$(1-x)^{-q}=\sum_{p=0}^{\infty}a_p x^p,\ a_p>0,$$

then for $|x_i y_j|<1$, we have (see [8])

$$\det[(1-x_i y_j)^{-q}]=\sum_{\substack{n\times n\\ l_1>l_2>\cdots>l_n\geq 0}} a_{l_1}a_{l_2}\cdots a_{l_n}\det|x_i^{l_i}|_{n\times n}\det|y_i^{l_j}|_{n\times n}.$$

Since when $0<x_1<x_2<\cdots<x_l, 0<y_1<y_2<\cdots<y_l$, the generalizad Vandermonde determinands[7]

$$\det(x_i^{l_j})_{n\times n}>0,\ \det(y_i^{l_j})_{n\times n}>0.$$

It may be observed that for $x_i>0$, $y_i>0$, we have

$$\det|(1-x_i y_j)^{-q}|_1^n>0,$$

so the lemma follows.

Using Lemma 4.1 and Rolle's theorem, we obtain

Theorem 4 If $a_{ij}<x_j$ (or $>x_{j+1}$, $j=0,1,\cdots n-1$), then the solution of the interpolation problem (20) with (21) exists and is uniquely determined.

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References

[1] Chui, C.K. and Smith, P.W., *Jour. Approx. Theory*, 13(1975), 421—430.

[2] T.N.E. Greville, Theory and Applications of spline functions, Acad. Press, New York (1968).

[3] Jerome, J.W., *SIAM Jour Numer. Anal.*, Vol. 10(1973), 808—819, 820—830.

[4] Lee, E.H. and Forsythe, G.E., *SIAM. Rew.*, 15(1973), 120—133.

[5] Shaback, R., *Jour. Approx. Theory*, 7(1973), 281—292.

[6] Shaback, R., *Jour. Approx. Theory*, 8(1973), 173—178.

[7] Ф.Р. Гантмахер, Теория Матриц, Москва (1953).

[8] 华罗庚，多复变函数论中的典型域的调和分析，北京(1958)。