A Simple Proof of L.Lovász Theorem

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In 1983 J. A. Bondy mentioned a result of L.Lovász. He said that it seems "magic". Now we give a simple proof of this result by using the basic probability theory only, and we may see why this result holds clearly.

Let G be a graph with a specified vertex r. A path $P=v_1v_2\cdots v_k$ is rooted at r if $v_1=r$. For $1\leqslant j\leqslant k$, the number of edges of G which are incident with v_j but with no v_i, $i<j$, is denoted by $f_j(P)$. A path $P=v_1v_2\cdots v_k$ rooted at r is maximal if $f_k(P)=0$, and $v(P)$ the number of vertices of P. L. Lovász has obtained the following

Theorem (L. Lovász, 1981) Let G be a graph with a specified vertex r. Then

$$\sum_P \prod_{j=1}^{v(P)-1} f_j(P)^{-1}=1.$$

Where the sum extends over all maximal paths P rooted at r.

Proof Let $\mathscr{F}=\{(v_j\to v_{j+1})P \mid P\in\mathscr{A}, v_j, v_{j+1}\in P, v_j \text{ is incident with } v_{j+1}\}$ be a sample space, where $\mathscr{A}$ is the set of maximal paths rooted at r. Define the probability on $\mathscr{F}$ as follows

$$p((v_j\to v_{j+1})P)=f_j(P)^{-1}.$$

Then the probability of occurrence of P should be

$$p(P)=\prod_{j=1}^{v(P)-1} f_j(P)^{-1}.$$

For all P, $Q\in\mathscr{A}$ $(P\neq Q)$, P, Q are incompatible and in any cose we can always reach a vertex v form r, which is the end-vertex of a path $P\in\mathscr{A}$. Then

$$1=\sum_P p(P)=\sum_P \prod_{j=1}^{v(P)-1} f_j(P)^{-1}.$$

Reference

J. A. Bondy, The lecture of the reconstruction of graphs.