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Generalized Inverses of a Partitioned Matrix*

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Let $A\{i,j,\cdots,k\}$ be the set of matrices $X=A^{(i,j,\cdots,k)}$ which satisfy equations $\langle i\rangle,\langle j\rangle,\cdots,\langle k\rangle$ from among $\langle 1\rangle$ $AXA=A$, $\langle 2\rangle$ $XAX=X$, $\langle 3\rangle$ $(AX)^*=AX$, $\langle 4\rangle$ $(XA)^*=XA$. G^* denotes the conjugate transpose of G, $A^+=A^{(1,2,3,4)}$. $\mathscr{R}(A)$ and $\mathscr{N}(A^*)$ denote the range of A and the null space of A^*, respectively. In this paper, the formula for computing inverse $[A(A_0B_1)]^+$ is presented, i.e.,

$$[A(A_0B_1)]^+=\begin{bmatrix} A^+(I-A_0\beta_1)(I-B_1\tilde{J}^+) \\ \begin{pmatrix}\beta_1(I-B_1\tilde{J}^+)\\ \tilde{J}^+\end{pmatrix}\end{bmatrix}. \qquad (1_a)$$

Where $\mathscr{R}(A_0)\subseteq\mathscr{R}(A)$, $\mathscr{R}(A)\cap\mathscr{R}(B_1)=\{0\}$, $\tilde{J}=(I-AA^+)(I-A_0\beta_1)B_1$, $\beta_1=(I+A_0^*A^{*+}A^+A_0)^{-1}A_0^*A^{*+}A^+$.

Now we put $B=(A_0B_1)$ in (1_a), then $\{0\}\subseteq\mathscr{R}(A)\cap\mathscr{R}(B)\subseteq\mathscr{R}(B)$. Specially, we have

$$(AB)^+=\begin{bmatrix} A^+(I-B\beta) \\ \beta\end{bmatrix}. \qquad (1_b)$$

Where,
$$\beta=\begin{cases}(I+B^*A^{*+}A^+B)^{-1}B^*A^{*+}A^+; & \mathscr{R}(A)\cap\mathscr{R}(B)=\mathscr{R}(B) \qquad (1'_b)\\ [(I-AA^+)B]^+. & \mathscr{R}(A)\cap\mathscr{R}(B)=\{0\} \qquad (1''_b)\end{cases}$$

Using formula (1_b), we obtain the expressions for $M^+=\begin{bmatrix}A & B\\ C & O\end{bmatrix}^+$ described in theorems 1—3 below, which simplify and improve the results in [1]、[3], and the result in theorem 4 is extremely useful to geodetic surveying[4].

Theorem 1 If $\mathscr{R}(AT)\cap\mathscr{R}(B)=\{0\}$, $\mathscr{R}(A^*U)\cap\mathscr{R}(C^*)=\{0\}$, then

$$M^+=\begin{pmatrix}D^+ & C^-\\ B^- & -B^+A(I-D^+A)C^+\end{pmatrix},$$

where $T=I-C^+C, U=I-BB^+$, $D=UAT$; $B^-=B^+(I-AD^+)\in B\{1,2,4\}, C^-=(I-D^+A)C^+ \in C\{1,2,3\}$.

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Theorem 2 If $\mathscr{R}(AT)\subseteq\mathscr{R}(U)$, $\mathscr{R}(A^*U)\subseteq\mathscr{R}(T)$, then

$$M^+=\begin{pmatrix}(UA)^+ & C^+\\ B^+ & -B^+AC^+\end{pmatrix}.$$

We denote the class of m×n matrices of rank r by $\mathscr{C}_r^{m\times n}$.

Theorem 3 Let $A\in\mathscr{C}_r^{m\times n}$, $B\in\mathscr{C}_{m-r}^{m\times(m-r)}$, $C\in\mathscr{C}_{n-r}^{(n-r)\times n}$, and $\mathscr{R}(A)\cap\mathscr{R}(B)=\{0\}$, $\mathscr{R}(A^*)\cap\mathscr{R}(C^*)=\{0\}$, then matrix M is nonsingular, and

$$M^{-1}=\begin{pmatrix}A^- & C^-\\ B^- & O\end{pmatrix}.$$

Where $A^-=D^+\in A\{1,2\}$.

Theorem 4 Let the submatrix O in M be $(n-r)\times(m-r)$ zero matrix, $A\in\mathscr{C}_r^{m\times n}$. If $\mathscr{R}(B)=\mathscr{N}(A^*)$, $\mathscr{R}(C^*)=\mathscr{N}(A)$, then matrix M is nonsingular, and

$$M^{-1}=\begin{pmatrix}A^+ & C^+\\ B^+ & O\end{pmatrix}.$$

Turning to the proof the formulae (1_a)、(1_b). Given $K=(AB)$, we deal separately with three cases the calculations of K^+, namely, (Ⅰ) $\mathscr{R}(A)\cap\mathscr{R}(B)=\mathscr{R}(B)$, (Ⅱ) $\mathscr{R}(A)\cap\mathscr{R}(B)=\{0\}$, (Ⅲ) $\{0\}\subseteq\mathscr{R}(A)\cap\mathscr{R}(B)\subseteq\mathscr{R}(B)$. For case (Ⅰ), we have $J=(I-AA^+)B=O$, suppose not, there exists a vector h_0 such that $Bh_0\neq AA^+Bh_0$, i. e., $Bh_0\overline{\in}\mathscr{R}(A)\cap\mathscr{R}(B)$, which is impossible since $\mathscr{R}(A)\cap\mathscr{R}(B)=\mathscr{R}(B)$. From this case, we see that B_1 and $\tilde{J}$ do not occur, and then, $A_0=B_1$, $\beta_1=(I+B^*A^{*+}A^+B)^{-1}B^*A^{*+}A^+$, moreover, $(1_b')$ holds and (1_a) becomes (1_b); for case (Ⅱ), we have $J\neq0$, suppose not, $\forall h$, $Bh=AA^+Bh\in\mathscr{R}(A)\cap\mathscr{R}(B)=\{0\}$, which is impossible since $B\neq0$. From this case, we see that A_0 and β_1 do not occur, and then, $B_1=B$, $\tilde{J}=J$, moreover, $(1_b'')$ holds and (1_a) becomes (1_b).

Before proving (1_a), we discuss the special cases (Ⅰ)、(Ⅱ). Let $K^+=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, then

$$A^+KK^+=A^+(A\alpha+B\beta). \tag{2}$$

Because $\mathscr{N}(A^+)=\mathscr{N}(A^*)\supseteq\mathscr{N}(K^*)=\mathscr{R}(I-KK^+)$, the equality $A^+(I-KK^+)=0$ holds, i.e., $A^+=A^+KK^+$; furthermore, because $\mathscr{R}(K^+)=\mathscr{R}(K^*)$, it is known that $\mathscr{R}(\alpha)=\mathscr{R}(A^*)=\mathscr{R}(A^+)=\mathscr{N}(I-A^+A)$, i. e., $\alpha=A^+A\alpha$. Substituting these into (2), we get $\alpha=A^+(I-B\beta)$. Next we determine β.

We observe that $KK^+=A\alpha+B\beta=A[A^+(I-B\beta)]+B\beta=AA^++J\beta$, $J\beta$ is clearly Hermitian, and there exists F such that $\beta=FJ^*$, i. e., $KK^+=AA^++JFJ^*$. According to $(AB)=K=KK^+K=(AA^++JFJ^*)(AB)$, we obtain $B=(AA^++JFJ^*)B$, so $J=JFJ^*B$ or $J=JFJ^*J$.

If $J\neq 0$, then we have $\beta=FJ^*=J^{(1)}$. Because of matrix K^+K is Hermitian, its submatrices $J^{(1)}A=\beta A=(\alpha B)^*=[A^+(I-B\beta)B]^*$; moreover, from the submatrices in both sides of equality $K=KK^+K$, we see that $B=AA^+(I-BJ^{(1)})B+BJ^{(1)}B$ or $AA^+B(I-BJ^{(1)})=B(I-J^{(1)}B)$, i. e., $\forall h$, $A^+AB(I-J^{(1)}B)h=B(I-J^{(1)}B)h\in\mathscr{H}(A)\cap\mathscr{H}(B)=\{0\}$, therefore, $(I-BJ^{(1)})B=0$. To sum up, $J^{(1)}$ satisfy $J^{(1)}A=0$. But $J^+A=0$ since $A^+J=0$, noticing that the uniqueness of $(AB)^+$, we must take $J^{(1)}=J^+$.

The equality $J^+A=0$ shows that K^+K is a block diagonal matrix.

If $J=0$, then $\mathscr{N}(K^*)=\mathscr{N}(A^*)$ holds since $K^*=\begin{bmatrix} I \\ B^*A^{*+}\end{bmatrix}A^*$, and hence, $\mathscr{N}(\beta)\supseteq\mathscr{N}(K^+)=\mathscr{N}(K^*)=\mathscr{N}(A^*)=\mathscr{H}(I-AA^+)$, i. e., $\beta(I-AA^+)=0$. Putting $d=A^+B$, it follows from the submatrices of K^+K that $\beta B=(\beta B)^*$, therefore, $\beta A=(\alpha B)^*=[A^+(I-B\beta)B]^*=[(A^+-d\beta)B]^*=(I-\beta B)d^*$, and hence $\beta B=\beta AA^+B=(I-\beta B)d^*d$ or $(I-\beta B)(I+d^*d)=I$, then we get $\beta=\beta AA^+=(I+d^*d)^{-1}d^*A^+$, i. e., $(1_b')$ holds.

Finally, we apply (1_b) and $(1_b'')$ to matrix $K=[(AA_0)B_1]$, and apply (1_b) and $(1_b')$ to matrix (AA_0), formula (1_a) is proved.

Furthermore, we can prove some properties of J^+:

(i) $\mathscr{N}(B)=\mathscr{N}(J)$ (so $\mathscr{H}(B^*)=\mathscr{H}(J^*)$); (ii) $J^+\in B\{1,2,4\}$;

(iii) $A^+(I-BJ^+)\in A\{1,2,4\}$; (iv) $(AA^++JJ^+)^+B=B$.

Hence a part of the contents in the present paper contain the main results of [1].

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