第4卷 第2期
1984年4月

数学研究与评论
JOURNAL OF MATHEMATICAL
RESEARCH AND EXPOSITION

Vol.4 No.2
April 1984

New Inverse Series Relations for Finite and Infinite Series with Applications*

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Abstract

Some new series inversion formulas of the general form

$$F(n)=\sum_{k=0}^{r}A_{k,n}f(n-mk)$$

if and only if

$$f(n)=\sum_{k=0}^{r}B_{k,n}F(n-mk)$$

valid for either $r=[n/m]$ or $r=\infty$ are presented. These relations generalize many of those given by the author in a long series of preceding papers. An interesting example is given by

$$A_{k,n}=(-y)^{k}A_{k}(p-\lambda n,t(1+\lambda m))$$

and

$$B_{k,n}=y^{k}A_{k}(p-\lambda n,(1-t)(1+\lambda m)),$$

where $A_k(a,b)=a\binom{a+bk}{k}/(a+bk)$ in terms of binomial coefficients. Here p,t,y and λ are arbitrary complex numbers. A corresponding Abel coefficient case occurs which uses numbers of the form $a(a+bi)^{i-1}/i!$. An application to special functions studied by Singhal and Kumari is given, and it is also shown that

$$\sum_{k=0}^{\infty}z^{k}A_{k}(a+ck,b)=x^{a}\frac{x-b(x-1)}{x-(b+c)(x-1)},$$

where $z=(x-1)x^{-b-c}$, with a corresponding case for the Abel coefficients

$$\sum_{k=0}^{\infty}z^{k}B_{k}(a+ck,b)=x^{a}\frac{1-b\log x}{1-(b+c)\log x},\quad \text{where } z=(\log x)x^{-b-c}$$

From these expansions we then have easily the new convolution formula for Rothe coefficients

*Received July 10, 1983.

$$\sum_{k=0}^{\infty} A_k(a+ck,b)A_{n-k}(d,b+c)=A_n(a+d+cn,b).$$

The same formula holds for Abel coefficients, i.e.

$$\sum_{k=0}^{n} B_k(a+ck,b)B_{n-k}(d,b+c)=B_n(a+d+cn,b).$$

For the most part this paper improves on the results obtained by the author in a series of papers starting in 1956.

1. Introduction. Twenty years ago the author [4] proved the new series inversion pair

$$F(n)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{a+bk}{n}f(k) \tag{1.1}$$

if and only if

$$\binom{a+bn}{n}f(n)=\sum_{k=0}^{n}(-1)^k\frac{a+bk-k}{a+bn-k}\binom{a+bn-k}{n-k}F(k), \tag{1.2}$$

together with the formal power series transform

$$\sum_{k=0}^{\infty}\binom{a+bk}{k}z^kf(k)=x^a\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-1}{x}\right)^nF(n), \tag{1.3}$$

where $z=(x-1)/x^b$. In subsequent papers [5],[6],[7],[8],[9],[11],[12], variations and extensions were given, together with applications to special functions and derivation of combinatorial identities.

In [5] it was shown that

$$F(a)=\sum_{k=0}^{r}(-)^kA_k(a,b)f(a+bk-k) \tag{1.4}$$

if and only if

$$f(a)=\sum_{k=0}^{r}\binom{a}{k}F(a+bk-k), \tag{1.5}$$

where the A's have been called by me [1],[2] Rothe coefficients and

$$A_k(a,b)=\frac{a}{a+bk}\binom{a+bk}{k}, \tag{1.6}$$

and the inversion pair had two valid values of r: (i) for arbitrary a and b, $r=\infty$; (ii) for a positive and $b\leqslant 0$ a negative integer, then $r=[a/(1-b)]$ is also valid.

The more general infinite series inversion

$$F(a)=\sum_{k=0}^{\infty}(-1)^kA_k(a,tb)f(a+bk-k) \tag{1.7}$$

if and only if

$$f(a)=\sum_{k=0}^{\infty}A_k(a,(1-t)b)F(a+bk-k) \tag{1.8}$$

was also noted in [5] wherein the parameter t was introduced, but a corresponding case with finite limits was not noted.

In [5] it was also found that (1.7) - (1.8) pass over by a limiting process to an Abel inversion pair

$$F(a) = \sum_{k=0}^{\infty} (-1)^k B_k(a, tb) f(a+bk) \tag{1.9}$$

if and only if

$$f(a) = \sum_{k=0}^{\infty} B_k(a, (1-t)b) F(a+bk), \tag{1.10}$$

where the Abel coefficients [1],[2] are defined by

$$B_k(a,b) = a(a+bk)^{k-1}/k!, \tag{1.11}$$

but again it was only noted in a few special cases that there is a corresponding finite case;

In [6] some variations of (1.4)—(1.5) were applied to the study of some special functions, but only when $r=\infty$.

In [7] the general inversion pair

$$F(n) = \sum_{k=0}^{[n/m]} \binom{p-n+mk}{k} f(n-mk) \tag{1.12}$$

if and only if

$$f(n) = \sum_{k=0}^{[n/m]} (-1)^k \binom{p-n+k}{k} \frac{p-n+mk}{p-n+K} F(n-mk) \tag{1.13}$$

was proved and used to study generalized Humbert polynomials. However, no corresponding infinite series inversion was noted.

Riordan [13] gives an extensive summary of the inversions I have developed, but almost exclusively he focuses on the instances with infinite upper summation limits.

Recently Singhal and Kumari [14] have presented a slight extension of (1.12)—(1.13), which they use to study functions somewhat more general than those studied in [7]. Their inversion pair introduces an interesting parameter λ as follows:

$$F(n) = \sum_{k=0}^{[n/m]} y^k \binom{p-\lambda n+\lambda mk}{k} f(n-mk) \tag{1.14}$$

if and only if

$$f(n) = \sum_{k=0}^{[n/m]} (-y)^k \binom{p-\lambda n+k}{k} \frac{p+\lambda mk-\lambda n}{p-\lambda n+k} F(n-mk), \tag{1.15}$$

p and λ being arbitrary complex numbers;

Singhal and Kumari remark that "for positive integral values of λ this extension is quite trivial but for arbitrary values of λ it is not so." Actually the proof for arbitrary λ is quite elementary, as we shall see below in our generalization. Perhaps the simplicity was obscured in [7] by a proof that was not so elegant.

The formula of Singhal and Kumari, with its parameter λ and the formulas of type (1.7)—(1.8) and (1.9)—(1.10) with the parameter t, suggest an overall generalization which it is the purpose of this paper to present and discuss, along with some new convolution identities.

2. A general lemma. Let us suppose that we have the orthogonality relations

$$\sum_{k=0}^{j} A_{k,n} B_{j-k,n-mk} = \binom{0}{j} = \delta_j^0 = \begin{cases} 0, & j \neq 0 \\ 1, & j = 0 \end{cases} \tag{2.1}$$

and

$$\sum_{k=0}^{j} B_{k,n} A_{j-k,n-mk} = \delta_j^0, \tag{2.2}$$

for some arrays of numbers $A_{k,n}$ and $B_{k,n}$. Then we have the general inverse pair

$$F(n) = \sum_{k=0}^{r(n)} A_{k,n} f(n-mk) \tag{2.3}$$

if and only if

$$f(n) = \sum_{k=0}^{r(n)} B_{k,n} F(n-mk), \tag{2.4}$$

and these will be valid in two cases: (i) when $r(n)=\infty$; (ii) when $r(n) = \left[\dfrac{n}{m}\right]$, $n \geqslant 0$ and $m = 1, 2, 3, \cdots$.

Proof. In the finite case we need to note that $r(n-mk) = r(n) - k$, which is what makes the bracket function so useful here. To show that (2.4) implies (2.3) we proceed as follows:

$$\begin{aligned}
\sum_{k=0}^{r(n)} A_{k,n} f(n-mk) &= \sum_{k=0}^{r(n)} A_{k,n} \sum_{j=0}^{r(n-mk)} B_{j,n-mk} F(n-mk-mj) \\
&= \sum_{k=0}^{r(n)} A_{k,n} \sum_{j=k}^{r(n)} B_{j-k,n-mk} F(n-mj) \\
&= \sum_{j=0}^{r(n)} F(n-mj) \sum_{k=0}^{j} A_{k,n} B_{j-k,n-mk} = \sum_{j=0}^{r(n)} F(n-mj)\delta_j^0 = F(n).
\end{aligned}$$

For the case $r=\infty$, the proof runs as follows:

$$\begin{aligned}
\sum_{k=0}^{\infty} A_{k,n} f(n-mk) &= \sum_{k=0}^{\infty} A_{k,n} \sum_{j=0}^{\infty} B_{j,n-mk} F(n-mk-mj) \\
&= \sum_{k=0}^{\infty} A_{k,n} \sum_{j=k}^{\infty} B_{j-k,n-mk} F(n-mj) \\
&= \sum_{j=0}^{\infty} F(n-mj) \sum_{k=0}^{j} A_{k,n} B_{j-k,n-mk} = \sum_{j=0}^{\infty} F(n-mj)\delta_j^0 = F(n).
\end{aligned}$$

The proofs that (2.3) implies (2.4) are the same upon interchanging A and B.

Application of the general lemma then merely requires to exhibit suitable $A's$ and $B's$ satisfying (2.1)—(2.2).

3. Generalized Rothe and Abel pairs. Since it was shown in [5] that

$$(3.1)\qquad \sum_{k=0}^{j}(-1)^k A_k(a,tb)\;A_{j-k}(a+bk-k,\;(1-t)b)=\binom{0}{j},$$

and

$$(3.2)\qquad \sum_{k=0}^{j}(-1)^k B_k(a,tb)B_{n-k}(a+bk,\;(1-t)b)=\binom{0}{j},$$

with A defined by (1.6) and B defined by (1.11), then (1.7)—(1.8) and (1.9)—(1.10) may be extended at once. In fact we have

$$(3.3)\qquad F(a)=\sum_{k=0}^{r(a)}(-y)^k A_k(a,tb)f(a+bk-k)$$

if and only if

$$(3.4)\qquad f(a)=\sum_{k=0}^{r(a)}y^k A_k(a,\;(1-t)b)F(a+bk-k),$$

valid as before for either $r=\infty$ in general, or for $r=[a/(1-b)]$ when $a>0$ and $b=0,-1,-2,\cdots$.

Analogously, the Abel extension is that

$$(3.5)\qquad F(a)=\sum_{k=0}^{r(a)}(-y)^k B_k(a,tb)f(a+bk)$$

if and only if

$$(3.6)\qquad f(a)=\sum_{k=0}^{r(a)}y^k B_k(a,\;(1-t)b)F(a+bk),$$

again valid for the two cases of $r(a)$.

4. Lambda extension of (3.3)—(3.4) and (3.5)—(3.6). The parameter λ of Singhal and Kumari may be introduced easily into our formulas. The reader should have no difficulty seeing that our general lemma (2.3)—(2.4) applies with

$$(4.1)\qquad A_{k,n}=(-y)^k A_k(\lambda n,\,t(1-\lambda m))$$

and

$$(4.2)\qquad B_{k,n}=y^k A_k(\lambda n,\;(1-t)(1-\lambda m)),$$

since

$$(4.3)\qquad \sum_{k=0}^{j}(-1)^k A_k(\lambda n,t\,(1-\lambda m))\;A_{j-k}\,(\lambda n-\lambda mk,\;(1-t)\,(1-\lambda m))=\binom{0}{j}$$

follows from (3.1) with $a=\lambda n$ and $b=1-\lambda m$.

Thus we have the Singhal- Kumari lambda-generalized Rothe inverse pair with parameter t

$$(4.4)\qquad F(n)=\sum_{k=0}^{r(n)}(-y)^k A_k(\lambda n, t(1-\lambda m))f(n-mk)$$

if and only if

$$(4.5)\qquad f(n)=\sum_{k=0}^{r(n)}y^k A_k(\lambda n, (1-t)(1-\lambda m))F(n-mk),$$

valid for the two values of $r(n)$ as before.

Again, since

$$(4.6)\qquad \sum_{k=0}^{j}(-1)^k B_k(\lambda n, -\lambda tm)B_{j-k}(\lambda n-\lambda mk, -\lambda(1-t)m)$$

follows from (3.2) with $a=\lambda n$ and $b=-\lambda m$, then we have the Singhal-Kumari lambda-generalized Abel inverse pair

$$(4.7)\qquad F(n)=\sum_{k=0}^{r(n)}(-y)^k B_k(\lambda n, -\lambda tm)f(n-mk)$$

if and only if

$$(4.8)\qquad f(n)=\sum_{k=0}^{r(n)}y^k B_k(\lambda n, -\lambda(1-t)m)F(n-mk)$$

valid for the two cases of $r(n)$.

5. Singhal-Kumari lambda-generalized Humbert inversion for finite and infinite series. The inversion formula pair (1.14)—(1.15) and the original case I established in [7] when $\lambda=1$ are not stated in really elegant form so as to exhibit the close alliance with the new pair (4.4)—(4.5) given above. But we may recast these by a simple transformation. It is easy to see that we may rewrite (1.14)—(1.15) first in the form

$$(p-\lambda n)F(n)=\sum_{k=0}^{r(n)}y^k A_k(p-\lambda n, \lambda m)(p-\lambda n+\lambda mk)f(n-mk)$$

if and only if

$$(p-\lambda n)f(n)=\sum_{k=0}^{r(n)}(-y)^k A_k(p-\lambda n, 1)(p-\lambda n+\lambda mk)F(n-mk).$$

Then letting $(p-\lambda n)F(n)=G(n)$ and $(p-\lambda n)f(n)=g(n)$, these become

$$(5.1)\qquad G(n)=\sum_{k=0}^{r(n)}y^k A_k(p-\lambda n, \lambda m)g(n-mk)$$

if and only if

$$(5.2)\qquad g(n)=\sum_{k=0}^{r(n)}(-y)^k A_k(p-\lambda n, 1)G(n-mk),$$

which is a more elegant way to state them and suggests that the t-parameter extension I introduced in [5] as in (4.4)—(4.8) can be carried through here also. This is indeed the case, and we have the new general Singhal-Kumari lambda-generalized t-parameter Humbert inversion pair for finite and infinite series:

$$G(n)=\sum_{k=0}^{r(n)}y^kA_k(p-\lambda n,(1-t)(1+\lambda m))g(n-mk) \tag{5.3}$$

if and only if

$$g(n)=\sum_{k=0}^{r(n)}(-y)^kA_k(p-\lambda n,t(1+\lambda m))G(n-mk) \tag{5.4}$$

valid for both the finite and infinite cases: (i) $r=\infty$; (ii) $r=[n/m]$.

The formal proof of (5.3)—(5.4) depends on the fact that

$$\sum_{k=0}^{j}(-1)^kA_k(p-\lambda n,t(1+\lambda m))A_{j-k}(p-\lambda n+\lambda mk,(1-t)(1+\lambda m))=\binom{0}{j}, \tag{5.5}$$

which follows from (3.1) with $a=p-\lambda n$ and $b=1+\lambda m$.

To retrieve the original Singhal-Kumari formula (1.14)—(1.15) from our new pair (5.3)-(5.4), set $t(1+\lambda m)=1$, so that $t=1/(1+\lambda m)$. Then $(1-t)(1+\lambda m)=\lambda m$, and (5.3)—(5.4) become

$$g(n)=\sum_{k=0}^{r}(-y)^kA_k(p-\lambda n,1)G(n-mk) \tag{5.6}$$

if and only if

$$G(n)=\sum_{k=0}^{r}y^kA_k(p-\lambda n,\lambda m)g(n-mk), \tag{5.7}$$

and of course then their formula was noted only for $r=[n/m]$.

The role of the parameter t in a coefficient such as $A_k(a,tb)$ is interesting. As t goes from 0 to 1 we may pass continuously from a binomial coefficient of form $\binom{a}{k}$ to the general Rothe type binomial coefficient $a\binom{a+bk}{k}\Big/(a+bk)$, and since in the general inversions we find t in one formula and $1-t$ in the other, we find a curious kind of symmetry in the formulas, best expressible for arbitrary t, but giving the same kind of coefficients in each formula only when $t=1/2$. Thus, almost exactly as noted in [5, p.398] but now valid for two upper limit values of r, we have the nice pair

$$F(a)=\sum_{k=0}^{r}(-1)^k\frac{a}{a+bk/2}\binom{a+bk/2}{k}f(a+bk-k) \tag{5.8}$$

if and only if

$$f(a)=\sum_{k=0}^{r}\frac{a}{a+bk/2}\binom{a+bk/2}{k}F(a+bk-k) \tag{5.9}$$

valid for $r=\infty$, or for $a>0$ and $r=[a/(1-b)], b=0,-1,-2,\cdots$.

6. Connection with special functions. In [6],[7],[14] and in many other papers on the subject, a special function is introduced as a coefficient in a generating function expansion, and then an inversion formula is used to represent an elementary function such as x^n as a linear combination of the special function terms.

Alternatively some orthogonality theorem does this for us. We may turn the game around, and use the series inversions we have developed here to invent special functions that will include all those previously studied as so many special cases. To see how this may work, consider that Singhal and Kumari defined $f_n^c(x,y,r,m)$ by the expansion

$$(6.1)\qquad (1+yt^m)^{-c}G(xt(1+yt^m)^{-r})=\sum_{n=0}^{\infty}t^n f_n^c(x,y,r,m),$$

where

$$(6.2)\qquad G(z)=\sum_{n=0}^{\infty}\gamma_n z^n,\quad \gamma_0\neq 0,$$

and found then that

$$(6.3)\qquad f_n^c(x,y,r,m)=\sum_{k=0}^{[n/m]}\binom{-c-nr+mrk}{k}y^k\gamma_{n-mk}x^{n-mk}.$$

The f_n^c polynomials extended the $P_n's$ discussed by me in [7] which in turn include many well-known polynomials. Singhal and Kumari then used (1.14)—(1.15) to invert (6.3) so as to obtain

$$(6.4)\qquad x^n=\gamma_n^{-1}\sum_{k=0}^{[n/m]}(-y)^k\binom{-c-nr+k}{k}\frac{mrk-nr-c}{k-nr-c}f_{n-mk}^c(x,y,r,m).$$

They then follow the idea used in [7] to express any polynomial in x as a linear combination of the polynomials f_n^c.

By means of (5.4) we find

$$(6.5)\qquad \sum_{n=0}^{\infty}g(n)u^n=\sum_{n=0}^{\infty}u^n\sum_{k=0}^{[n/m]}(-y)^kA_k(p-\lambda n,t(1+\lambda m))G(n-mk)$$
$$=\sum_{n=0}^{\infty}u^nG(n)\sum_{k=0}^{\infty}(-y)^kA_k(p-\lambda n-\lambda mk,t(1+\lambda m))u^{mk}.$$

Thus, in order to sum (6.5) we must find a closed form for a series of the form $\sum_{k=0}^{\infty}z^kA_k(a+ck,b)$, and since this does not seem to have appeared before, we shall solve the general case, which has many other applications to the discovery of identities.

We shall prove that

$$(6.6)\qquad \sum_{k=0}^{\infty}z^kA_k(a+ck,b)=x^a\cdot\frac{x-b(x-1)}{x-(b+c)(x-1)},\ \text{where } z=\frac{x-1}{x^{b+c}}.$$

Of course the case $c=0$ is very well known [1],[2].

As for the proof, we have by (1.6) and the special case $c=0$

$$\sum_{k=0}^{\infty} z^k A_k(a+ck,b) = \sum_{k=0}^{\infty} z^k \frac{a+ck}{a} A_k(a,c+b)$$

$$= \sum_{k=0}^{\infty} z^k A_k(a,c+b) + \frac{c}{a} z \sum_{k=0}^{\infty} k z^{k-1} A_k(a,c+b)$$

$$= \left(1+\frac{c}{a} z D_z\right) x^a,$$

where $z=(x-1)x^{-b-c}=x^a+\frac{c}{a}zD_x(x^a)\cdot D_z x = x^a + \frac{czx^{a-1}}{D_x z}$,

but
$$D_x z = \left(1-b-c+\frac{b+c}{x}\right)x^{-b-c},$$

and the final simplification is routine.

For application to (6.5) we take $a=p-\lambda n, b=t(1+\lambda m), c=-\lambda m$, and $z=-yu^m$. Thus we find

$$\text{(6.7)} \qquad \sum_{k=0}^{\infty}(-y)^k A_k(p-\lambda n-\lambda mk,\ t(1+\lambda m))u^{mk}$$
$$= x^{p-\lambda n}\cdot\frac{x-t(1+\lambda m)(x-1)}{x-(t+t\lambda m-\lambda m)(x-1)},$$

so that

$$\text{(6.8)} \qquad \sum_{n=0}^{\infty} g(n)u^n = x^p\cdot\frac{x-t(1+\lambda m)(x-1)}{x-(t+t\lambda m-\lambda m)(x-1)}\sum_{n=0}^{\infty} G(n)(ux^{-\lambda})^n,$$

which we may take as a generalization of (6.1) of Singhal and Kumari with $g(n)$ taking the place of their f_n^c. In some cases the right-hand side of (6.8) can be gotten in closed form so as to parallel their work more closely. For example, take $G(n)=1$ for all $n\geqslant 0$ and $t=1/(1+\lambda m)$. Then

$$\sum_{n=0}^{\infty} g(n)u^n = \frac{x^p}{(x\lambda m+1-\lambda m)(1-ux^{-\lambda})}.$$

We should also indicate next that our parallel to (6.3)—(6.4) may be stated in the following form:

$$\text{(6.9)} \qquad G(n)=F_n^p(x,y,m,t,\lambda)=\sum_{k=0}^{[n/m]} y^k A_k(p-\lambda n,\ (1-t)(1+\lambda m))x^{n-mk}g(n-mk).$$

Then

$$\text{(6.10)} \qquad x^n = \frac{1}{g(n)}\sum_{k=0}^{[n/m]}(-y)^k A_k(p-\lambda n, t(1+\lambda m))F_{n-mk}^p(x,y,m,t,\lambda).$$

In this form f_n^c is replaced by F_n^p with $p=-c$ and $g(n)$ takes the place of γ_n. A fuller account of these generalizations may best be taken up in a separate paper.

7. Corresponding results for Abel coefficients. Expansion (6.6) for Rothe coefficients has an analogue for the Abel coefficients (1.11). Since [1],[2]

$$\text{(7.1)} \qquad \sum_{k=0}^{\infty} B_k(a,b)z^k = x^a, \text{ where } z=\frac{\log x}{x^b},$$

it is not surprising that we may parallel the work in the previous section to find that

$$(7.2) \qquad \sum_{k=0}^{\infty} B_k(a+ck,b)\,z^k = x^a \cdot \frac{1-b\log x}{1-(b+c)\log x},$$

where now $z=(\log x)/x^{b+c}$.

Some of the steps in deriving these formal power series results are:

$$\sum_{k=0}^{\infty} B_k(a+ck,b)\,z^k = \left(1+\frac{c}{a}zD_z\right)x^a;$$

$$D_x z = \frac{1-(b+c)\log x}{x^{b+c+1}}.$$

It would be possible to use the Abel coefficients to obtain still another extension of the work of Singhal and Kumari.

8. New convolution identities. From (6.6) we have

$$\sum_{k=0}^{\infty} z^k A_k(a+ck,b) \cdot \sum_{k=0}^{\infty} z^k A_k(d+ck,b) = x^{a+d} \cdot \left(\frac{x-b(x-1)}{x-(b+c)(x-1)}\right)^2$$

$$= \sum_{k=0}^{\infty} z^k A_k(a+e+ck,b) \cdot \sum_{k=0}^{\infty} z^k A_k(d-e+ck,b),$$

whence we find the convolution identity

$$(8.1) \quad \sum_{k=0}^{n} A_k(a+ck,b)\,A_{n-k}(d+cn-ck,b) = \sum_{k=0}^{n} A_k(a+e+ck,b)\,A_{n-k}(d-e+cn-ck,b).$$

In the same way (7.2) yields

$$(8.2) \quad \sum_{k=0}^{n} B_k(a+ck,b)\,B_{n-k}(d+cn-ck,b) = \sum_{k=0}^{n} B_k(a+e+ck,b)\,B_{n-k}(d-e+cn-ck,b).$$

Because of the fact that our infinite series do not sum to just x^a, but $x^a R(x)$ with $R(x)$ a rational function of x (or $\log x$), it is not so easy to get closed forms for (8.1) and (8.2) as is the case when $c=0$. Thus we find no immediate easy extension of the know [1],[2] Rothe and Abel addition theorems

$$\sum_{k=0}^{n} A_k(a,b)\,A_{n-k}(d,b) = A_n(a+d,b)$$

and

$$\sum_{k=0}^{n} B_k(a,b)\,B_{n-k}(d,b) = B_n(a+d,b).$$

However, we may carry out an interesting experiment which allows us to cancel a denominator and a numerator in two of the R's, and we shall thereby be able to get a closed form for a slightly different convolution. On the one hand we have

$$(8.3)\qquad \sum_{n=0}^{\infty} z^n A_n(a+cn,b)=x^a\cdot\frac{x-b(x-1)}{x-(b+c)(x-1)},\quad z=\frac{x-1}{x^{b+c}},$$

and

$$(8.4)\qquad \sum_{n=0}^{\infty} w^n A_n(d+en,b+c)=x^d\cdot\frac{x-(b+c)(x-1)}{x-(b+c+e)(x-1)},\quad w=\frac{x-1}{x^{b+c+e}}.$$

When we multiply these two together, the numerator in (8.4) cancels the denominator in (8.3) and the result can be expanded by means of (6.6), so that we shall have

$$(8.5)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{n} z^k w^{n-k} A_k(a+ck,b)A_{n-k}(d+en-ek,b+c)$$

$$=x^{a+d}\frac{x-b(x-1)}{x-(b+c+e)(x-1)}=\sum_{n=0}^{\infty}u^n A_n(a+d+(c+e)n,b),$$

where now $u=(x-1)/x^{b+c+e}$.

In order to secure a combinatorial identity from this, we have to require that $z^k w^{n-k}=u^n$. But it is easily verified that

$$(8.6)\qquad z^k w^{n-k}=u^n(x^{-e})^k,$$

so that the only time we may deduce anything is when we make $e=0$. The result is that we have discovered the new convolution addition theorem for Rothe coefficients

$$(8.7)\qquad \sum_{k=0}^{n} A_k(a+ck,b)A_{n-k}(d,b+c)=A_n(a+d+cn,b).$$

In exactly parallel fashion, we may use (7.2) to show that the corresponding theorem for Abel coefficients is

$$(8.8)\qquad \sum_{k=0}^{n} B_k(a+ck,b)B_{n-k}(d,b+c)=B_n(a+d+cn,b).$$

Another observation about (6.6) is that by writing $c-b$ for c, we can say

$$(8.9)\qquad \sum_{n=0}^{\infty} z^n A_n(a+cn-bn,b)=x^a\frac{x-b(x-1)}{x-c(x-1)},\quad z=\frac{x-1}{x^c}$$

so that we find

$$(8.10)\qquad \left\{\sum_{n=0}^{\infty} z^n A_n(a+cn-bn,b)\right\}^{-1}=\sum_{n=0}^{\infty} w^n A_n(-a+bn-cn,c)$$

$$=x^{-a}\frac{x-c(x-1)}{x-b(x-1)},\quad \text{where } w=\frac{x-1}{x^b}.$$

Finally, to sum the series

$$(8.11)\qquad \sum_{k=0}^{n} A_k(a+ck,b)A_{n-k}(d+c(n-k),b)$$

we would have to find a clever way to expand

$$x^{a+c}\cdot\frac{[x-b(x-1)]^2}{[x-(b+c)(x-1)][x-(b+d)(x-1)]}, \tag{8.12}$$

but there does not appear to be an easy way to do this.

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