第4卷 第3期 数学研究与评论 Vol.4 No.3
1984年7月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION July 1984

Counting Rooted Planar Maps*

Liu Yanpei

Institute of Applied Mathematics, Academia Sinica, Beijing, China and Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3GI

Abstract

This is basically an expository paper on counting rooted planar maps as a part of the enumerative theory of planar maps which was founded by W.T. Tutte in the sixties.

However, several new results and a certain number of simplifications and provided.

§1 Introduction

The original paper of W.T. Tutte [13] on the enumerative theory of planar maps has brought about a series of papers on enumerating planar triangulations. In 1982, W.T. Tutte published his famous expository paper in the field to make the theory simpler. In the same period of time, the enumerations of general planar maps have also been investigated. A certain number of results have been obtained, although relatively fewer than triangulations. However, two types of elegant formulae obtained by Tutte should be mentioned: one determining the numbers of rooted general, nonseparable and 3-connected planar maps with the edge number given [16]; the other determining the number of rooted Eulerian planar maps with given vertex-partition according to the valencies of vertices [15].

In this paper, the main purpose is to show a quadratic functional equation, as Tutte's papers [16, 17] implied, of the enumerating function of rooted general planar maps such that the parametric expressions are derived directly for finding the explicit formulae of the numbers of combinatorially distinct rooted general, loopless, simple, non-separable, 2-connected, 3-connected and 2-nonseparable planar maps. Here, those about loopless, simple, 2-connected and 2-nonseparable are new. All the procedures of finding them are in some sense simplified.

*Received Aug. 4, 1983.

The terminology not explained here can be found in [6,7,8,16].

§2 Functional Equation

Let $\mathcal{M}$ be the set of all the rooted general planar maps. The function $p(x,y)$ of two independent variables x,y, denoted by

$$p(x,y)) = \sum_{M\in\mathcal{M}} x^{m(M)}y^{n(M)}, \tag{2.1}$$

is said to be the enumerating function with two indices m,n which represent the edge number and the valency of the outer face of M respectively.

$\mathcal{M}$ may be divided into three classes:

$$\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_I + \mathcal{M}_J, \tag{2.2}$$

where $\mathcal{M}_0$ consists of only one map, i.e. the vertex map; $\mathcal{M}_I$, $\mathcal{M}_J$ represent the subset of $\mathcal{M}$ in which all the maps have the root-edge separable, non-separable respectively.

Of course, the contribution of $\mathcal{M}_0$ to $p(x,y)$ is

$$1. \tag{2.3}$$

As to $\mathcal{M}_I$, for any $M\in\mathcal{M}_I$, let $M'=\mathrm{M}-\mathrm{A}$, i.e. the map obtained by deleting the root-edge A form M. Then

$$M' = M_1' + M_2', \tag{2.4}$$

where M_1', M_2' are components of M'. And M_1', M_2' may be rooted as taking the edge adjacent to A on the boundary of the outer face of M as the root-edge whose direction agrees with that on the boundary along A. Since both $M_1', M_2' \in\mathcal{M}$, the contribution of $\mathcal{M}_I$ to $p(x,y)$ is

$$xy^2(p(x,y))^2. \tag{2.5}$$

Finally, for any $M\in\mathcal{M}_J$, $M'=M-A$ may be rooted as taking the edge successive to A in the direction consistent with what is defined by A on the boundary of the outer face of M. Let l be the number of the edges on the boundary of the outer face of M' but not on that of M. Then, it can be seen that for any $M\in\mathcal{M}_J$, there exists an $M'\in\mathcal{M}$ with $l\geqslant 0$, and conversely, for any $M'\in\mathcal{M}$, there are just $n+1$ maps, i.e. those of $l=0,1,\cdots,n, n=n(M')$, in $\mathcal{M}_J$ corresponding to M'. In other words, the contribution of $\mathcal{M}_J$ to $p(x,y)$ is

$$\sum_{M\in\mathcal{M}}\sum_{l=0}^{n(M)} x^{m(M)+1}y^{n(M)-l+1}$$

$$= xy\sum_{M\in\mathcal{M}} x^{m(M)}\,\frac{y-y^{-n(M)}}{y-1}\,y^{n(M)},$$

or simply writing it as

$$\frac{xy}{y-1}(y\;p(x,y)-h(x)), \tag{2.6}$$

where

$$h(x)=p(x,1)=\sum_{M\in\mathcal{M}}x^{m(M)}. \tag{2.7}$$

Consequently, according to (2.3), (2.5) and (2.6), we obtain the functional equation satisfied by $p(x,y)$ as follows

$$p(x,y)=1+xy^2(p(x,y))^2+\frac{xy}{y-1}(y\ p(x,y)-h(x)). \tag{2.8}$$

§3 Parametric Expressions

First, we transform the functional equation (2.8) into the following form

$$(2x(y-1)y^2p(x,y)+xy^2-y+1)^2=\lambda(x,y), \tag{3.1}$$

where

$$\lambda(x,y)=(xy^2-y+1)^2-4y^2(y-1)x(y-1-xyh(x)). \tag{3.2}$$

If ξ is a power series of x such that

$$\xi(\xi-1)xp(x,\xi)+\frac{1}{2\xi}(x\xi^2-\xi+1)=0, \tag{3.3}$$

i.e. the perfect square on the left hand side of (3.1) equals zero, then we have

$$\begin{cases}\lambda(x,\xi)=0;\\ \left.\dfrac{\partial}{\partial y}\lambda(x,y)\right|_{y=\xi}=0.\end{cases} \tag{3.4}$$

According to (3.2), the following two equations with x and $h=h(x)$ as variables, ξ as a parameter, can be found

$$\begin{cases}\xi^4x^2-2\xi^2(\xi-1)(2\xi-1)x+(\xi-1)^2+4\xi^3(\xi-1)x^2h=0;\\ 2\xi^3x^2-(2\xi(\xi-1)+\xi^2+4\xi(\xi-1)(2\xi-1))x+(\xi-1)+(2\xi^3+6\xi^2(\xi-1))x^2h=0.\end{cases} \tag{3.5}$$

From both these equations, the parametric expressions of x and h can be derived.

§4 General planar Maps

In order to determine $h(x)$, the enumerating function of rooted general planar maps with the edge number as an index, we solve the equations in (3.5). By eliminating the terms with x^2h, we have

$$\begin{cases}2\xi^3x^2h=\xi^2(3\xi-2)x+(\xi-1)(\xi-2);\\ \xi^4x^2+2\xi^2(\xi-1)^2x+(\xi-1)^2(2\xi-3)=0.\end{cases} \tag{4.1}$$

From the last one of (4.1), we obtain

$$x=\frac{1}{\xi^2}(-(\xi-1)^2\pm(\xi-1)(\xi-2)). \tag{4.2}$$

Since $h(x)$ should be a power series of x with all the terms positive, only the negative sign is available whence we have

$$x=\frac{1}{\xi^2}(1-\xi)(2\xi-3). \tag{4.3}$$

According to the first equation of (4.1), we find

$$h=\frac{1}{2}\frac{\xi(1-\xi)(6\xi^2-14\xi+8)}{(1-\xi)^2(3-2\xi)^2}=\frac{\xi(4-3\xi)}{(3-2\xi)^2}. \tag{4.4}$$

If the substitution of parameters

$$\theta=\frac{1}{\xi} \tag{4.5}$$

is introduced, (4.3) becomes

$$x=(1-\theta)(3\theta-2). \tag{4.6}$$

From (4.4),

$$h=\frac{4\theta-3}{(3\theta-2)^2}. \tag{4.7}$$

By using Lagrange's Theorem, the following explicit form about $h(x)$ may be found [16].

$$h(x)=1+\sum_{m=1}^{\infty}\frac{2\cdot 3^m(2m)!}{(m+2)!m!}x^m. \tag{4.8}$$

§5 Loopless Planar Maps

A Loopless map, as the term suggests, is such a map in which there is no loop. Let $\mathcal{M}_{NL}$, $\mathcal{M}_{NL}(\cdot,2)$ be the set of all rooted loopless planar maps and such maps with the valency of the outer face being 2 respectively. Then we have the following conclusions:

Lemma 2.1 Let us write

$$\begin{cases}\mathcal{M}_{NL}(m)=\{M|M\in\mathcal{M}_{NL}\ \text{and}\ m(M)=m\};\\ \mathcal{M}_{NL}(m,2)=\{M|M\in\mathcal{M}_{NL}(\cdot,2)\ \text{and}\ m(M)=m\}.\end{cases} \tag{5.1}$$

Then we have

$$|\mathcal{M}_{NL}(m)|=|\mathcal{M}_{NL}(m+1,2)|, \tag{5.2}$$

where $|X|$ denotes the cardinality of set X.

Proof In fact, we may find a 1-to-1 correspondence between $\mathcal{M}_{NL}(m)$ and $\mathcal{M}_{NL}(m+1,2)$ for any $m\geqslant 0$. Apparently, when $m=0$, $|\mathcal{M}_{NL}(0)|=\mathcal{M}_{NL}(1,2)|$, i.e. the vertex map corresponds to the link map.

When $m\geqslant 1$, suppose that $M\in\mathcal{M}_{NL}(m+1,2)$, we take $M'=M-A$, of course, $M'\in\mathcal{M}_{NL}(m)$. Conversely, from $M'\in\mathcal{M}_{NL}(m)$, we may add a new edge connecting the two ends of the root-edge to M' such that the new edge and the root-edge form the boundary of the outer face of the resultant map M. Then $M\in\mathcal{M}_{NL}(m+1,2)$. Paying attention to the uniqueness of the above procedure, a 1-to-1 correspondence has been found. ∎

Accorging to the lemma, the following theorem may be obtained.

Theorem 2.1 Let $h^{NL}(x)$ and $g^{NL}(x)$ be the enumerating functions of $\mathcal{M}_{NL}$ and $\mathcal{M}_{NL}(\cdot,2)$ with the edge number as the index. Then we have

$$xh^{NL}(x)=g^{NL}(x). \tag{5.3}$$

Proof In fact,

$$g^{NL}(x) = \sum_{M \in \mathcal{M}_{NL}(\cdot, 2)} x^{m(M)}$$

$$= \sum_{M' \in \mathcal{M}_{NL}} x^{m(M')+1}$$

$$= x\, h^{NL}(x). \qquad \blacksquare$$

Based on the theorem and [8], we have

$$\begin{cases} x = \dfrac{t-1}{t^4}; \\ h^{NL}(x) = t^2(2-t). \end{cases} \tag{5.4}$$

By employing Lagrange's Theorem, the explicit expression about $h^{NL}(x)$ may be derived as

$$h^{NL}(x) = 1 + \sum_{m=1}^{\infty} \frac{6(4m+)!}{(3m+3)!\,m!} x^m. \tag{5.5}$$

It is of interest to note that the formula is just the same as counting rooted strict planar triangulations with the number of inner vertices as the index [13]. However, up to now, a 1-to-1 correspondence between them has not been found.

§6 Simple Planar Maps

By a simple map, we shall as usual mean that in a map, there are neither loops nor multi-edges. Let $\mathcal{M}_S$ be the set of all the rooted simple planar maps, $h^S(x)$ be the enumerating function of such maps with the edge number as the index. According to [8], we have

$$h^{NL}(x) = h^S(g^{NL}(x)). \tag{6.1}$$

The following parametric expressions have been found

$$\begin{cases} X = g^{NL} = \dfrac{(t-1)\,(2-t)}{t^2}; \\ h^S(X) = t^2(2-t) \end{cases} \tag{6.2}$$

Moreover, since

$$\frac{d}{dx} h^S = t(4-3t) \quad \frac{t^3}{(4-3t)} = t^4, \tag{6.3}$$

if we denote $G(x) = t^4$, then

$$h^S(x) = 1 + \int_0^x G(x)\,dx, \quad h^S(0) = 1. \tag{6.4}$$

In this case, by using Lagrange's Theorem, we obtain

$$G(x) = 1 + 4x + \sum_{m=2}^{\infty} \frac{x^m}{m!} \left\{ \sum_{i=0}^{m-1} \frac{4(2m+3)!\,(2m-i-2)!}{i!\,(m-i-1)!\,(2m-i+3)!} \right\}. \tag{6.5}$$

Hence, from (6.4), we find

$$h^S(x)=1+x+2x^2+\sum_{m=3}^{\infty}\frac{x^m}{m!}\left(\sum_{i=0}^{m-2}\frac{4(2m+1)!(2m-i-4)!}{i!(m-i-2)!(2m-i+1)!}\right). \tag{6.6}$$

§7 Non-separable Planar Maps

A non-separable map is such a map in which there is no vertex separable i.e. no cut-vertex. Let $\mathscr{M}_2$ be the set of all the rooted non-separable planar maps, and

$$h_2(x)=\sum_{M\in\mathscr{M}_2}x^{m(M)}, \tag{7.1}$$

the enumerating function of $\mathscr{M}_2$ with the edge number as the index.

However, from any $M\in\mathscr{M}_2$, the rooted general planar maps of a certain type may be found by embedding rooted general planar maps into some angles of M such that the root-vertices identify with the vertices of corresponding angles. Conversely, for any rooted general planar map, let the non-separable component including the root-edge be called its 2-nucieus, then the map may also be considered as the one obtained by the above procedure from the 2-nucleus as a map in $\mathscr{M}_2$.

Since the number of angles is twice the edge number, wa have

$$h(x)=h_2(x(h(x))^2). \tag{7.2}$$

On account of (4.6), (4.7), and introducing

$$\eta=\frac{1}{3\theta-2}, \tag{7.3}$$

we find

$$x=\frac{1}{3\eta^2}(\eta-1). \tag{7.4}$$

Furthermore, let us write

$$w=x(h(x))^2, \tag{7.5}$$

then the following are derived

$$\begin{cases} w=\dfrac{1}{27}(\eta-1)(4-\eta)^2; \\ h_2(w)=\dfrac{\eta}{3}(4-\eta). \end{cases} \tag{7.6}$$

Consequently, we may obtain

$$h_2(x)=1+\sum_{m=1}^{\infty}\frac{2(3m-3)!}{(2m-1)!m!}x^m, \tag{7.7}$$

by using Lagrange's Theorem [16].

§8 2-connected Planar Maps

By a 2-connected map, we shall mean that it is non-separable and simple. Let $\mathscr{M}_2^S$ be the set of all the rooted 2-connected planar maps and

$$h_2^S(x)=\sum_{M\in\mathscr{M}_2^S}x^{m(M)} \tag{8.1}$$

be the enumerating function of $\mathscr{M}_2^S$ with the edge number as the index.

Since any rooted non-separable planar map may be considered as a map obtained from a rooted 2-connected planar map by an assignment of edges for which some rooted non-separable ones with the valency of outer faces being 2 are substituted except for the one in which there is only one edge and the edge is a loop. It can be seen that different rooted non-separable planar maps correspond to different rooted 2-connected ones or different assignment of edges; of course, any rooted 2-connected planar map itself is a rooted non-separable planar map, we have

$$h_2(x) - x = h_2^S(g_2(x)), \tag{8.2}$$

where $g_2(x)$ is the enumerating function of the rooted non-separable planar maps with the outer face valency being 2.

Accorging to the reasoning of theorem 2.1, with the exception of the edge number less than 3, which are easily determined directly, we may find

$$g_2(x) = x + x^2 + x(h_2(x) - 1 - 2x). \tag{8.3}$$

Let us write

$$u = g_2(x) = x(h_2(x) - x). \tag{8.4}$$

From (7.6), we derive

$$\begin{cases} u = 3^{-6}(\eta - 1)(4 - \eta)^3(\eta + 2)^2; \\ h_2^S(u) = 3^{-3}(4 - \eta)(\eta + 2)^2. \end{cases} \tag{8.5}$$

If the following substitution is introduced

$$\varphi = \frac{4 - \eta}{\eta + 2}, \tag{8.6}$$

then we have

$$\begin{cases} u = 2^5(1 - \varphi)\dfrac{\varphi^3}{(1 + \varphi)^6}; \\ h_2^S(u) = 2^3\dfrac{\varphi}{(\varphi + 1)^3}. \end{cases} \tag{8.7}$$

In conclusion, by using Lagrange's Theorem, we may deduce the following expression in terms of definite integrals.

$$h_2^S(x) = 1 + x + \sum_{m=2}^{\infty} \frac{(6m - 4)!\, J(m)}{(3m - 1)!\,(2m - 2)!} \frac{x^m}{m!}, \tag{8.8}$$

where

$$J(m) = 4m\int_0^1 y^{3m-1}(1 - y)^{2m-2}(2y - 1)^{m-2}dy - (3m - 1)\int_0^1 y^{3m-2}(1 - y)^{2m-2}(2y - 1)^{m-2}dy. \tag{8.9}$$

Therefore, an explicit or recursive formula may be found by calculating the integrals directly or recursively. Unfortunately, it does not seem to be as simple as the others in the paper.

§9 3-connected Planar Maps

A map is said to be a 3-connected map if it is 2-connected, with at least 4 vertices, and can not be separated into at least 2 components each of which has at least 2 edges by the removal of any two vertices. In [16], such maps are called c-nets also. From the definition, any 3-connected map has no multi-edges.

Let $\mathcal{M}_3$ be the set of all the rooted 3-connected planar maps and

$$h_3(x)=\sum_{M\in\mathcal{M}_3}x^{m(M)}\tag{9.1}$$

be the enumerating function of $\mathcal{M}_3$ with the edge number as the index.

According to Tutte's theory [16], we have

$$h_3(x)=x^2-\frac{2x}{1+x}-zx,\tag{9.2}$$

where

$$\begin{cases}x=-\dfrac{1}{27}\eta(3+\eta)^2;\\ z=-\eta\dfrac{3+2\eta}{(3+\eta)^2}.\end{cases}\tag{9.3}$$

If the following substitution is introduced

$$s=\frac{3+2\eta}{3+\eta},\tag{9.4}$$

then we obtain

$$\begin{cases}x=(1-s)s;\\ z=\dfrac{1-s}{(2-s)^3}.\end{cases}\tag{9.5}$$

Let us denote

$$-z=\sum_{m=1}^{\infty}R_mx^m.\tag{9.6}$$

In order to determine R_m more effectively, Tutte provided an elegant recursive formula [16] by using Lagrange's Theorem although an explicit formula can be found directly. However, we here provide a simpler recursive one for calculating R_m as

$$\begin{cases}R_m=\dfrac{7m-22}{2m}R_{m-1}+\dfrac{2m-1}{m}R_{m-2},\ m\geqslant 3;\\ R_1=-1,\ R_2=2,\end{cases}\tag{9.7}$$

without using Lagrange's Theorem, because it can be shown that $-z=y$ satisfies the following differential equation

$$(4x^2+7x-2)\frac{dy}{dx}+(6x-15)y=2.\tag{9.8}$$

Consequently, from (9.2), we have

$$h_3(x)=\sum_{m=4}^{\infty}((-1)^m2+R_{m-1})x^m.\tag{9.9}$$

§10 2-nonseparable Planar Maps

By a 2-nonseparable map is meant that it is nonseparable and no two vertices can separate it into at least 2 components, each of which has at least one vertex by removing the two vertices and the edges incident to them. Naturally, in such map, multi-edges are allowed, and it is easily seen that any 3-connected map is 2-nonseparable. Conversely, any 2-nonseparable map would be 3-connected whenever multi-edges were not considered, except for the trivial cases of the edge number less then 4.

Let $\mathscr{M}_2^N$ be the set of all the rooted 2-nonseparable planar maps and

$$h_2^N(x) = \sum_{M \in \mathscr{M}_2^N} x^{m(M)} \tag{10.1}$$

be the enumerating function of $\mathscr{M}_2^N$ with the edge number as the index.

According to what we hrve just discussed, the following functional equation may be found

$$h_2^N(x) = \frac{1}{1-x} + \left(\frac{x}{1-x}\right)^3 + h_3\left(\frac{x}{1-x}\right)^3. \tag{10.2}$$

Since we have

$$\frac{1}{(1-x)^k} = \sum_{i=0}^{\infty} \frac{(k+i-1)!}{(k-1)!\,i!} x, \quad k \geqslant 1, \tag{10.3}$$

from (9.9) and (10.2), the following formula may be obtained

$$h_2^N(x) = \sum_{m=0}^{\infty} x^m + \sum_{m=3}^{\infty} \binom{m-1}{2} x^m + \sum_{m=4}^{\infty} \sum_{i=4}^{m} \binom{m-1}{i-1} a_i x^m = \sum_{m=0}^{\infty} \beta_m x^m, \tag{10.4}$$

where

$$\beta_m = \begin{cases} 1, & 0 \leqslant m \leqslant 2; \\ 2, & m = 3; \\ \binom{m-3}{2} + \sum_{i=4}^{m} \binom{m-1}{i-1} R_{i-1}, & m \geqslant 4, \end{cases} \tag{10.5}$$

in which the following identity has been used:

$$\sum_{i=4}^{m} \binom{m-1}{i-1} (-1)^i = \binom{m-2}{2}, \tag{10.6}$$

and R_i, $i \geqslant 2$, is given by (9.7).

Now, the aim we posed in §1 has been achieved.

The author's sincere thanks are given to Professor W.T. Tutte for his direction on the subject and his encouragement to write such an expository paper.

References

[1] Brown, W.G., Enumeration of triangulations and the disk. *proc. London Math. Soc.* (3) 14 (1964), 746—748.

[2] Brown, W.G., Enumeration of non-separable planar maps. *Canad.* J *Math.* XV (1963) 526—545.

[3] Brown, W G. and Tutte, W. T., On the enumeration of rooted nonseparable planar maps, *Canad.* J. *Math.*. Vol. 16 (1964), 572—577.

[4] Cori, R., Planar maps and bracketing systems, in Combinatorial Structures and their Applica tions. R. Guy et al. eds. 1970, 35—36.

[5] Goulden, I.P., and Jackson, D.M., Combinatorial Enumeration. John Wiley G Sons. 1983.

[6] Liu Yanpei, Enumeration of rooted separable planar maps, Research Repors CORR 82—47. University of Waterloo. 1982.

[7] Liu Yanpei. Enumerating rooted loopless planar maps. Researcb Report CORR 83—4, University of Waterloo. 1983.

[8] Liu Yanpei, Enumerating rooted simple planar maps. Research Report CORR 83—9. University of Waterloo. 1983.

[9] Mullin, R. C., Enumeration of rooted triangular maps. *Amer. Math. Monthly*, 71 (1964). 1007-1010.

[10] Mullin, R.C., On counting rooted triangular maps. *Canad.* J. *Math.* 17 (1965), 373—382.

[11] Mullin, R.C., and Schellenberg, P. J., The enumeration of c-nets via quadrangulations, J, *Combinatorial Theory*. Vol. 4 (1968). 259—276.

[12] Mullin, R,C., Nemeth, E, and Schellenberg, P.J., The enumeration of almost cubic maps, Graph Theory and Computing. Baton Rouge. R,C, Mullin et al, eds, 1970. 281—295.

[13] Tutte, W.T., A census of planar triangulations, *Canad*, J, *Math*, Vol. 14 (1962). 21—38.

[14] Tutte, W.T., A census of Hamiltonian polygons, *Canad*, J, *Math*, 14 (1982). 402—417.

[15] Tutte, W.T., A census of slicing, *Canad.* J. *Math.* 14 (1982), 708—722.

[16] Tutte, W.T., A census of planar maps, *Canad.* J. *Math.* Vol. 15 (1963), 249—271.

[17] Tutte, W.T., On the enumeration of planar maps, *Bull. Amer. Math. Soc.* 74(1968), 64—74.

[18] Tutte, W.T., Counting planar maps, J. *Recreational Math.* 1(1968), 19—27.

[19] Tutte, W.T., The enumerative theory of planar maps, in "A Survey of Combinatorial Theory", J.N. Srivastava et al. eds. North-Holland, 1973, 437—448.

[20] Tutte, W.T., On elementary calculus and the Good formula, J. *Combinatorial. Theory*, Vol. 18 (1975), 99—137.

[22] Tutte, W.T., On the enumeration of convex polyhedra, J. *Combinatorial Theory*, (B) 28 (1980), 105—126.

[22] Tutte, W. T., Counting rooted triangulations, *Annals of Discrete Mathematics* 12 (1982), 243—253.

[23] Wittaker, E. T., and Watson, G.N., A Course of Modern Analysis, Cambridge, 1952.