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An Unified Approach to Approximation Theorems of Dini-Lipschitz-Type*

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Abstract

This survey paper studies the approximation of (polynomial) processes for which the operator norms do not form a bounded sequence. In view of familiar direct estimates and quantitative uniform boundedness principles, a unified approach is given to results concerning the equivalence of Dini-Lipschitz-type conditions with (strong) convergence on (smoothness) classes. Emphasis is laid upon the necessity of these conditions, essential ingredients of the proofs are suitable modifications of the familiar gliding hump method. Apart from the classical results concerned with Fourier partial sums, explicit applications are treated for (trigonometric as well as algebraic) Lagrange interpolation, interpolatory quadrature rules based upon Jacobi knots, multipliers of strong convergence, and for Bochner-Riesz means of multivariate Fourier series for parameter values below the critical index.

1 Introduction

Let $C_{2\pi}$ be the Banach space of 2π-periodic, continuous functions on the real axis $\mathbf{R}$, endowed with the usual sup-norm $\|\cdot\|_C$. Concerning the uniform convergence of the Fourier series

$$(1.1)\qquad f(x)\sim\sum_{k=-\infty}^{\infty}f^{\wedge}(k)e^{ikx},\qquad f^{\wedge}(k):=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(u)e^{-iku}du,$$

of $f\in C_{2\pi}$, that is, the uniform convergence of the nth partial sum

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$$(1.2)\qquad (S_n f)(x) := \sum_{k=-n}^{n} f^{\wedge}(k) e^{ikx}$$

towards f, a sufficient criterion is provided by the classical Dini-Lipschitz condition (cf. [3, p. 105; 36I, p. 63])

$$(1.3)\qquad \omega_1(1/n, f; C_{2\pi}) \log n = o_f(1) \qquad (n \to \infty),$$

where the (first order) modulus of continuity of f is given by

$$(1.4)\qquad \omega_1(t, f; C_{2\pi}) := \sup_{|h|<t} \|f(u+h) - f(u)\|_C .$$

It was already shown by Faber and Lebesgue in 1910 (see [11, p. 408ff; 14], also [32; 36I, p. 302]) that conditions of type (1.3) are in fact necessary if the individual function f is replaced by a whole (generalized) Lipschitz class. More precisely, if ω is a continuous function on $[0, \infty)$ such that

$$(1.5)\qquad 0 = \omega(0) < \omega(s) \leq \omega(t), \quad \omega(t)/t \leq \omega(s)/s \qquad (0 < s \leq t)$$

(abstract modulus of continuity, cf. [31, p. 96ff]), then the Dini-Lipschitz-type condition

$$(1.6)\qquad \omega(1/n) \log n = o(1) \qquad (n \to \infty)$$

is necessary and sufficient for the uniform convergence

$$(1.7)\qquad \|S_n f - f\|_C = o_f(1) \qquad (n \to \infty)$$

to take place for each $f \in C_{2\pi}$ satisfying the (generalized) Lipschitz condition

$$(1.8)\qquad \omega_1(t, f; C_{2\pi}) = O_f(\omega(t)) \qquad (t \to 0+).$$

Starting from this classical result, it is the purpose of the present paper to indicate a general approach to the subject. On the basis of quantitative uniform boundedness principles, previously developed in [4-6; 9], Section 2 delivers (quantitative) equivalence assertions of the Dini-Lipschitz type for quite a general class of operators in Banach spaces. Emphasis is laid upon the necessity of the conditions, essential ingredients of the proofs are given by suitable modifications of the familiar gliding humpmethod. Section 3 is devoted to some first applications to (trigonometric as well as algebraic) Lagrange interpolation, interpolatory quadrature rules based on Jacobi knots, multipliers of strong convergence, and to Bochner-Riesz means of multivariate Fourier series for parameter values below the critical index. See also the introduction to Section 3, and for some historical comments the end of Section 3. 4, 6.

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2 General Theory

Let X be a (complex) Banach space (with norm $\|\cdot\|_X$), and X^* be the class of functionals T on X which are sublinear, i. e.,

$$|T(f+g)|\leqslant|Tf|+|Tg|,\quad |T(af)|=|a|\,|Tf|$$

for all $f,g\in X$ and $a\in \mathbf{C}$ (: = set of complex numbers), and which are bounded, i. e.,

$$\|T\|_{X^*}:=\sup\ \{|Tf|;\|f\|_X=1\}<\infty.$$

For a normed linear space Y let $[X,Y]$ be the space of bounded, linear operators of X into Y (for short $[X]:=[X,X]$). Let $\{\varphi_n\}$ be a sequence of positive numbers satisfying ($n\in \mathbf{N}$: = set of natural numbers)

(2.1) $$0<\varphi_{n+1}<\varphi_n,\quad \lim_{n\to\infty}\varphi_n=0,$$

and ω a function satisfying (1.5) as well as

(2.2) $$\lim_{t\to 0+}\omega(t)/t=\infty.$$

Theorem 2.1. *Let $T_n\in X^*$ and $U_n,U\in[X,Y]$ satisfy* (cf. (2.20-22))

(2.3) $$\|U_nf-Uf\|_Y\leqslant C_1\|U_n\|_{[X,Y]}\,|T_nf|\qquad (f\in X).$$

Moreover, suppose that there exist a sequence $\{\varphi_n\}$, subject to (2.1), *as well as elements $g_n\in X$ and constants $C>O$ such that* ($j,n\in \mathbf{N}$)

(2.4) $$\|g_n\|_X\leqslant C_2,$$

(2.5) $$|T_ng_j|\leqslant C_3\ \min\ \{1,\varphi_n/\varphi_j\},$$

(2.6) $$\|U_ng_n\|_Y\geqslant C_4\|U_n\|_{[X,Y]}.$$

Then for each ω satisfying (1.5), (2.2) *and each (strictly, positive sequence $\{\psi_n\}$ with*

(2.7) $$\omega(\varphi_n)=o(\psi_n)\qquad\qquad (n\to\infty)$$

the Dini-Lipschitz-type condition

(2.8) $$\|U_n\|_{[X,Y]}\,\omega(\varphi_n)=o(\psi_n)$$

is necessary and sufficient for

(2.9) $$\|U_nf-Uf\|_Y=o_f(\psi_n)\quad\text{on}\quad \{f\in X;\,|T_nf|=O_f(\omega(\varphi_n))\}.$$

Proof Obviously, (2.8) implies (2.9) in view of (2.3), whereas the necessity is a consequence of the quantitative uniform boundedness principles, given in [4; 5; 6, Chapter 2; 9]. Let us include a proof, for the sake of completeness.

Assume then that (2.8) does not hold, i. e.,

(2.10) $$\|U_n\|_{[X,Y]}\,\omega(\varphi_n)/\psi_n\geqslant C_5>0$$

for infinitely many $n\in \mathbf{N}$. Starting with an arbitrary $n_1\in \mathbf{N}$ one may successively construct a monotonely increasing subsequence $\{n_k\}\subset \mathbf{N}$ satisfying (2.10) as well as ($k\geqslant 2$)

(2.11) $$\omega(\varphi_{n_k})\leqslant\min\left\{\frac{1}{2}\omega(\varphi_{n_{k-1}}),\ \frac{\psi_{n_k}}{k},\ \frac{\psi_{n_{k-1}}}{C_2(k-1)\|U_{n_{k-1}}\|_{[X,Y]}}\right\},$$

(2.21) $$\sum_{j=1}^{k-1}\omega(\varphi_{n_j})/\varphi_{n_j}\leqslant\omega(\varphi_{n_k})/\varphi_{n_k},$$

(2.13) $$\|U_{n_k}f_{k-1}\quad Uf_{k-1}\|_Y\leqslant\frac{\psi_{n_k}}{k},\quad f_{k-1}:=\sum_{j=1}^{k-1}\omega(\varphi_{n_j})g_{n_j}.$$

Indeed, (2.11,12) will be satisfied in view of (1.5), (2.1,2,7), whereas (2.13) is a consequence of (1.5), (2.5,9). Since X is complete and (cf. (2.4,11))

$$\sum_{j=k}^{\infty}\omega(\varphi_{n_j})\|g_{n_j}\|_X\leqslant 2C_2\omega(\varphi_{n_k}), \tag{2.14}$$

the case $k=1$ implies that $f_0:=\sum_{j=1}^{\infty}\omega(\varphi_{n_j})g_{n_j}$ is well-defined as an element in X. Then, given $n\geqslant n_1$, let $k\in\mathbf{N}$ be such that $n_k\leqslant n\leqslant n_{k+1}$. By (1.5), (2.1,5,11,12) one has

$$\begin{aligned}|T_nf_0|&\leqslant C_3\varphi_n\sum_{j=1}^{k}\omega(\varphi_{n_j})/\varphi_{n_j}+C_3\sum_{j=k+1}^{\infty}\omega(\varphi_{n_j})\\&\leqslant 2C_3\varphi_n\omega(\varphi_{n_k})/\varphi_{n_k}+2C_3\omega(\varphi_{n_{k+1}})\leqslant 4C_3\omega(\varphi_n),\end{aligned}$$

so that (2.9) implies

$$\|U_nf_0-Uf_0\|_Y=o(\psi_n). \tag{2.15}$$

On the other hand, if $U_{n_k}-U$ is applied to f_0, this yields by (2.4, 6, 10, 11, 13, 14)

$$\begin{aligned}\|U_{n_k}f_0-Uf_0\|_Y&\geqslant\omega(\varphi_{n_k})(\|U_{n_k}g_{n_k}\|_Y-\|Ug_{n_k}\|_Y)\\&\quad-\|(U_{n_k}-U)f_{k-1}\|_Y-\|(U_{n_k}-U)(f_0-f_k)\|_Y\\&\geqslant\psi_{n_k}[C_4C_5-3(C_2\|U\|_{[X,Y]}+1)/k],\end{aligned}$$

which is a contradiction to (2.15), proving the theorem. ∎

Obviously, one may replace the parameter n with $n\to\infty$ by a continuous $\rho>o$ with $\rho\to\infty$ (cf. Section 3.2). Moreover, (2.3) is actually not used for the necessity, that is, for (2.9) ⇒ (2.8).

Note that an appropriate choice of $\{\psi_n\}$ does not only determine necessary bounds for ω to ensure convergence (choose $\psi_n=1$), but also regains previous results on the sharpness of (2.3). In fact, if one chooses $\psi_n=\|U_n\|_{[X,Y]}\omega(\varphi_n)$, then (2.8) is trivially violated, so that necessarily there exists an element $f_\omega\in X$ with

$$\begin{aligned}&|T_nf_\omega|=O(\omega(\varphi_n)),\\&\|U_nf_\omega-Uf_\omega\|_Y\neq o(\|U_n\|_{[X,Y]}\omega(\varphi_n)).\end{aligned} \tag{2.16}$$

Let us mention that this interpretation has indeed many applications concerning the sharpness of error bounds in various areas of analysis. For details see [1; 4-9; 9].

Analogously to Theorem 2.1, one may formulate a large-O-version in the sense that small-o-rates in (2.7-9) are replaced by large-O-one's. In fact, a proof may then be given via a reduction to the classical uniform boundedness principle (cf. [6, p. 21]).

Theorem 2.2. *If the conditions of Theorem 2.1, i. e., (2.3-6), hold true; then for each ω, $\{\psi_n\}$ satisfying (1.5) and*

$$\omega(\varphi_n)=O(\psi_n)\qquad(n\to\infty) \tag{2.17}$$

the Dini-Lipschitz-type condition

$$\|U_n\|_{[X,Y]}\omega(\varphi_n)=O(\psi_n) \tag{2.18}$$

is necessary and sufficient for

(2.19) $$\|U_nf-Uf\|_Y=O_f(\psi_n)\quad on\{f\in X;\ |T_nf|=O_f(\omega(\varphi_n))\}.$$

Proof Since $T_n\in X^*$, the space

$$z:=\{f\in X;\ |T_nf|=O_f(\omega(\varphi_n)),n\to\infty\},$$

$$\|f\|_z:=\|f\|_X+\sup_{n\in N}|T_nf|/\omega(\varphi_n),$$

is a Banach space. Now (2.19) implies

$$\|(U_n-U)f\|_Y=O_f(\psi_n)\quad (f\in Z),$$

thus by the classical uniform boundedness principle

$$\|U_n-U\|_{[Z,Y]}=O(\psi_n).$$

Moreover, by (1.5), (2.1,5)

$$|T_ng_j|\leqslant C_3\min\{1,\varphi_n/\varphi_j\}\leqslant C_3\min\{1,\omega(\varphi_n)/\omega(\varphi_j)\},$$

so that by (2.4)

$$\|g_j\|_Z\leqslant C_2+C_3/\omega(\varphi_j)\leqslant A_1/\omega(\varphi_j).$$

Therefore in view of (2.4,6,17) one obtains

$$\begin{aligned}\|U_n\|_{[X,Y]}\omega(\varphi_n)&\leqslant A_2\|U_ng_n\|_Y\omega(\varphi_n)\\&\leqslant A_3\|U_ng_n-Ug_n\|_Y/\|g_n\|_Z+A_2\|U\|_{[X,Y]}\|g_n\|_X\omega(\varphi_n)\\&\leqslant A_3\|U_n-U\|_{[Z,Y]}+O(\omega(\varphi_n))=O(\psi_n),\end{aligned}$$

thus (2.18). Of course, the other implication follows by (2.3). ∎

In the applications the process $\{T_n\}$ mainly serves as a measure of smoothness. For example, T_n may be given in terms of moduli of continuity or Peetre K-functionals. Yet another concretization of T_n is performed via the error of best approximation

$$E(f,M_n):=E_X(f,M_n):=\inf\{\|f-p_n\|_X;\,p_n\in M_n\},$$

where $M_n, n\in N$, are certain linear manifolds with $M_n\subset M_{n+1}\subset X$. In this connection a special class of operators U_n is of interest, namely those for which

(2.20) $$U_nP_n=UP_n\qquad (P_n\in M_n),$$

(2.21) $$\|U\|_{[X,Y]}\leqslant C\|U_n\|_{[X,Y]}.$$

Indeed, then one has (cf. (3.2))

(2.22) $$\begin{aligned}\|U_nf-Uf\|_Y&=\inf_{p_n\in M_n}\|(U_n-U)(f-p_n)\|_Y\\&\leqslant(\|U_n\|_{[X,Y]}+\|U\|_{[X,Y]})\inf_{p_n\in M_n}\|f-p_n\|_Y\\&\leqslant C\|U_n\|_{[X,Y]}E(f,M_n)\end{aligned}$$

thus an estimate of type (2.3). In this situation there often exist operators (de la Vallée Poussin means, Fejér-Hermite processes) $V_n\in[X]$ such that

(2.23) $$\|V_n\|_{[X]}\leqslant C,\qquad V_n(X)\subset M_{2n},\qquad U_nV_n=U_n.$$

Then Theorem 2.1 ⟨2.2⟩ delivers (terms in brackets ⟨…⟩ give the large-O-version as a conclusion of Theorem 2.2).

Corollary 2.3 *Let* $\{\varepsilon_n\}$, *subject to* (2.1), *and* $\{\psi_n\}$ *be such that*

(2.24) $\varepsilon_n = O(\varepsilon_{2n})$, $\psi_n > 0$,

$$\varepsilon_n = o(\psi_n) \qquad \langle \cdots = O(\psi_n) \rangle.$$

Suppose that for U_n, $U \in [X,Y]$ with (2.20, 21) there exist $V_n \in [X]$ with (2.23). Then the following two assertions are equivalent:

(i) $\|U_n\|_{[X,Y]}\varepsilon_n = o(\psi_n) \qquad \langle \cdots = O(\psi_n) \rangle$,

(ii) $\|U_n f - Uf\|_Y = o_f(\psi_n) \qquad \langle \cdots = O_f(\psi_n) \rangle$

for each $f \in X$ with $E(f, M_n) = O_f(\varepsilon_n)$.

Proof By the definition of an operator norm there exists $f_n \in X$ with

(2.25) $$\|f_n\|_X = 1, \quad \|U_n f_n\|_Y \geqslant \frac{1}{2}\|U_n\|_{[X,Y]}.$$

To apply Theorem 2.1 ⟨2.2⟩ to

$$g_n = V_n f_n, \qquad T_n f = E(f, M_n),$$

one has (2.3,,4,6) in view of (2.22,23,25). Moreover, (2.5) follows with $\varphi_n = \varepsilon_n^2$ since by (2.23,24)

$$T_n g_j = 0 \leqslant \min\{1, \varepsilon_n^2/\varepsilon_j^2\} \qquad (2j \leqslant n),$$

$$T_n g_j \leqslant \|g_j\|_X \leqslant C \leqslant C^* \min\{1, \varepsilon_n^2/\varepsilon_j^2\} \quad (2j > n).$$

Finally. (2.7) ⟨(2.17)⟩ is valid for $\omega(t) = t^{1/2}$ by (2.24). ∎

Note that the proof still works if the error of best approximation is replaced by functionals $T_n \in X^*$ for which, apart from (2.3), a condition of type

(2.26) $$|T_n p_m| \leqslant C\|p_m\|_X \min\{1, \varphi_n/\varphi_m\} \qquad (p_m \in M_m)$$

holds true. In many situations, the latter estimate follows by an iterative application of a Jackson- and a Bernstein-type inequality. See, for example, (3.3) where T_n is the modulus of continuity (1.4).

3 Applications

Typical for the application of an equivalence assertion of Dini-Lipschitz-type is a situation given via (2.20-22) where the operator norms of the process $\{U_n\}$ do not form a bounded sequence, that is, where

(3.1) $$\limsup_{n\to\infty} \|U_n\|_{[X,Y]} = \infty.$$

In fact, the results of Section 2 then provide minimal (smoothness) properties the elements should satisfy in order to ensure convergence. From this point of view note that the case $\|U_n\|_{[X,Y]} = O(1)$ is already governed by the classical Banach-Steinhaus theorem.

3.1 Fourier Partial Sums

For the partial sums (1.2) an application of Theorem 2.1 yields

Corollary 3.1 *Let ω satisfy (1.5), (2.2). The following two assertions are equivalent $(n \to \infty)$:*

(i) $\|S_n\|_{[C,\pi]}\omega(1/n)=o(1)$,

(ii) $\|S_nf-f\|_C=o_f(1)$ on $\{f\in C_{2\pi};\ \omega_1(1/n,f;C_{2\pi})=O_f(\omega(1/n))\}$.

Proof Choose $X=Y=C_{2\pi}$, $U_n=S_n$, $U=I$ (: = the identity), and $T_nf=\omega_1(1/n,f;C_{2\pi})$. If Π_n denotes the set of trigonometric polynomials of degree n, then (2.3) follows in view of

$$\|S_nf-f\|_C\leqslant(1+\|S_n\|_{[C_{2\pi}]})E(f,\Pi_n)\leqslant C(1+\|S_n\|_{[C_{2\pi}]})\omega_1(1/n,f;C_{2\pi}),\tag{3.2}$$

the latter estimate being a consequence of Jackson's theorem (cf. [3, p.97]). Conversely, let $f_n\in C_{2\pi}$ be such that

$$\|f_n\|_C=1,\quad \|S_nf_n\|_C\geqslant\tfrac{1}{2}\|S_n\|_{[C,\pi]},$$

and consider the classical delayed means $V_n=(1/n)\ \sum_{K=n+1}^{2n}S_k$ of de La Vallée Poussin. Since (2.23) holds true for these means (with $M_n=\Pi_n$, cf. [3,p.108]), the elements $g_n=V_nf_n\in\Pi_{2n}$ satisfy (2.4,6) as well as (2.5) with $\varphi_n=1/n$, since by the mean value theorem and Bernstein's inequality (cf.[3, p.99])

$$T_ng_j=\sup_{|h|<1/n}\|g_j(u+h)-g_j(u)\|_C\leqslant(1/n)\|g_j'\|_C\leqslant Cj/n.\tag{3.3}$$

Therefore an application of Theorem 2.1 with $\psi_n=1$, $n\in N$, completes the proof. ∎

Since the operator norms $\|S_n\|_{[C,\pi]}$ behave like $\log n$ (cf. [3, p.42]), Corollary 3.1 indeed regains the result, mentioned in Section 1, namely the the equivalence of (1.6) and (1.7) for the class (1.8). Moreover, the proof given above also shows that the assumptions of Corollary 2.3 hold true. Therefore

Corollary 3.2. *Let $\{\varepsilon_n\}$, subject to (2.1), and $\{\psi_n\}$ satisfy (2.24). The following assertions are equivalent:*

(i) $\|S_n\|_{[C,\pi]}\varepsilon_n=o(\psi_n)$ $\qquad \langle\ldots=O(\psi_n)\rangle$,

(ii) $\|S_nf-f\|_C=o_f(\psi_n)$ $\qquad \langle\ldots=O_f(\psi_n)\rangle$

for each $f\in C_{2\pi}$ with $E(f,\Pi_n)=O_f(\varepsilon_n)$.

In fact, all the arguments given above remain true if the Banach space $C_{2\pi}$ is replaced by e.g. $L^1_{2\pi}$, the space of 2π-periodic, Lebesgue integrable functions with norm $\|f\|_1:=(1/2\pi)\int_{-\pi}^{\pi}|f(u)|\,du$. Since again $\|S_n\|_{[L^1_{2\pi}]}\sim\log n$, a parallel application of Theorem 2.1 also delivers

Corollary 3.3. *Let ω satisfy (1.5), (2.2) The following assertions are equivalent:*

(i) $\omega(1/n)\log n=o(1)$,

(ii) $\|S_nf-f\|_1=o_f(1)$ on $\{f\in L^1_{2\pi};\ \omega_1(1/n,f;L^1_{2\pi})=O_f(\omega(1/n))\}$.

3.2 Bochner-Riesz Means

Let R^N, $N\in \mathbf{N}$, be the N-dimensional Euclidean space with inner product $xy:=\Sigma_{i=1}^N x_iy_i$ and norm $|x|:=(xx)^{1/2}$, and let Z^N be the N-fold Cartesian product of Z (: = set of integers). With $Q^N := \{x \in \mathbf{R}^N;\ -\pi \leqslant x_j < \pi,\ 1 \leqslant j \leqslant N\}$ let $C_{2\pi}(Q^N)$ denote the space of continuous functions on R^N, 2π-periodic in each variable, with the usual sup-norm $\|\cdot\|_C$. For the Bochner-Riesz means ($\alpha \geqslant 0$, $\rho > 0$, $x \in \mathbf{R}^N$, $f \in C_{2\pi}(Q^N)$)

$$(B^\alpha_\rho f)(x) := \sum_{K \in Z^N} b_{\rho,\alpha}(k) f^\wedge(k) e^{ikx}, \quad b_{\rho,\alpha}(k) := \left(\max\left\{1 - \frac{|k|^2}{\rho^2},\ 0\right\}\right)^\alpha$$

$$f^\wedge(k) := (2\pi)^{-N} \int_{Q^N} f(u) e^{-iku} du,$$

one has the direct estimate

$$\|B^\alpha_\rho f - f\|_C \leqslant C \|B^\alpha_\rho\|_{[C_{2\pi}(Q^N)]}\, \omega_2(\rho^{-1}, f; C_{2\pi}(Q^N)), \tag{3.4}$$

$$\omega_2(t, f; C_{2\pi}(Q^N)) := \sup_{0 < |h| < t} \|f(x) - 2f(x+h) + f(x+2h)\|_C, \tag{3.5}$$

thus an inequality of type (2.3). Indeed, oen may, for example, apply the local divisibility argument of [8, Section 4] to the multiplier $(b_{\rho,\alpha}(k) - 1/\|B^\alpha_\beta\|_{[C_{2\pi}(Q^N)]}$ and use the equivalence of $\omega_2(t,f;C_{2\pi}(Q^N))$ with the Peetre k-functional $k(t^2, f; C_{2\pi}(Q^N), C^{(2)}_{2\pi}(Q^N))$ (cf. [2, p.258]).

An application of the results of Section 2 then yields

Corollary 3.4. *Let $\alpha \geqslant 0$. For each ω satisfying (1.5), (2.2) ⟨(1.5)⟩ and each (strictly) positive function ψ on $(0, \infty)$ wihe $(t \to 0+)$*

$$\omega(t) = o(\psi(t^{-1})) \qquad \langle \cdots = O(\psi(t^{-1}))\rangle \tag{3.6}$$

the following assertions are equivalent $(\rho \to \infty)$:

(i) $\|B^\alpha_\rho\|_{[C_{2\pi}(Q^N)]}\, \omega(\rho^{-2}) = o(\psi(\rho^2)) \qquad \langle \cdots = O(\psi(\rho^2))\rangle$,

(ii) $\|B^\alpha_\rho f - f\|_C = o_f(\psi(\rho^2)) \qquad \langle \cdots = O_f(\psi(\rho^2))\rangle$

for each $f \in C_{2\pi}(Q^N)$ with $\omega_2(t, f; C_{2\pi}(Q^N)) = O_f(\omega(t^2))$.

Proof For an infinitely often differentiable function λ on $[0, \infty)$ satisfying

$$0 \leqslant \lambda(t) \leqslant 1, \qquad \lambda(t) = \begin{cases} 1, & 0 \leqslant t \leqslant 1 \\ 0, & t \geqslant 2, \end{cases} \tag{3.7}$$

let generalized de la Vallée Poussin means be given by

$$(L_\rho f)(x) := \sum_{k \in Z^N} \lambda(|k|/\rho) f^\wedge(k) e^{ikx}.$$

These delayed means satisfy (2.23), *i.e.* (cf.[21;22] and the literature cited there),

$$\|L_\rho\|_{[C_{2\pi}(Q^N)]} \leqslant C, \quad L_\rho f \in \Pi^N_{2\rho}, \quad B^\alpha_\rho L_\rho = B^\alpha_\rho, \tag{3.8}$$

where Π^N_ρ denotes the set of trigonometric polynomials of radial degree at most $\rho > 0$. Let $f_\rho \in C_{2\pi}(Q^N)$ be such that

$$\|f_\rho\|_C = 1, \qquad \|B^\alpha_\rho f_\rho\|_C \geqslant \tfrac{1}{2} \|B^\alpha_\rho\|_{[C_{2\pi}(Q^N)]}$$

and apply Theorem 2.1 ⟨2.2⟩ to

$$X = Y = C_{2\pi}(Q^N), \quad U_\rho = B^\alpha_\rho, \quad U = I,$$

$$T_\rho f = \omega_2(\rho^{-1}, f; C_{2\pi}(Q^N)), \quad g_\rho = L_\rho f_\rho.$$

Then (2.3,4,6) follow by (3.4,8). Moreover, corresponding to (3.3) one has (cf [2, p.257])

$$T_{\rho_1} f_{\rho_1} \leqslant \rho_2^{-2} \sum \left\| \frac{\partial^2}{\partial x_1^{j_1} \cdots \partial x_N^{j_N}} g_{\rho_1} \right\|_C \leqslant C \rho_2^{-2} \rho_1^2,$$

the sum being extended over all multiindices $(j_1, \cdots, j_N)$ with $j_1 + \cdots + j_N = 2$. Thus (2.5) follows with $\varphi_\rho = \rho^{-2}$, and the proof is complete. ∎

Note that (cf. (3.1)) for $0 \leqslant a \leqslant (N-1)/2$ the operator norms $\|B_\rho^a\|$ are not bounded as $\rho \to \infty$ (cf. [27,p.170ff]) so that the present corollary then gives a necessary and sufficient condition for convergence (set $\psi(\rho) = 1$ for $\rho = 0$). On the other hand, the equivalence also holds true for $a > (N-1)/2$, where the norms are bounded. Of course, then Corollary 3.4 (ii)$\Rightarrow$(i), as applied to the case $\psi(t^{-1}) = \omega(t)$ yields the sharpness of the direct estimate (3.4) (cf. (2.16)).

Let us finally mention that again the verification of the conditions of Theorem 2.1 ⟨2.2⟩ proceeds completely parallel to the one given above, if $C_{2\pi}(Q^N)$ is replaced by the Banach space $L_{2\pi}^p(Q^N)$, $1 \leqslant p < \infty$, of 2π-periodic functions, *pth* power integrable over Q^N.

3.3 Trigonometric Lagrange Interpolation

For $f \in C_{2\pi}(= C_{2\pi}(Q))$ the trigonometric Lagrange polynomials, interpolating f at the equidistant knots $x_{kn} = 2k\pi/(2n+1)$, $0 \leqslant k \leqslant 2n$, are given by

$$(A_n f)(x) := \frac{1}{2n+1} \sum_{k=0}^{2n} f(x_{kn}) D_n(x - x_{kn}),$$

$$D_n(x) := 1 + 2\sum_{k=1}^{n} \cos kx = \frac{\sin(n+1/2)x}{\sin x/2}.$$

It is well-known that these operators are projections on Π_n, their norms being unbounded (they behave like $\log n$; for details see [11,p.424], also [20,p.365ff, 392, 492ff; 36II,p. 1ff]).

Corollary 3.5. *Let* $\{\varepsilon_n\}$, *subject to* (2.1), *and* $\{\psi_n\}$ *satisfy* (2.24). *The following assertions are equivalent:*

(i) $\quad \|A_n\|_{[C_{2\pi}]} \varepsilon_n = o(\psi_n) \qquad \langle \cdots = O(\psi_n) \rangle,$

(ii) $\quad \|A_n f - f\|_C = o_f(\psi_n) \qquad \langle \cdots = O_f(\psi_n) \rangle$

for each $f \in C_{2\pi}$ *with* $E(f, \Pi_n) = O_f(\varepsilon_n)$.

Proof The result follows as an application of Corollary 2.3 upon setting

$$X = Y = C_{2\pi},\ M_n = \Pi_n,\ U_n = A_n,\ U = I,\ V_n = J_n$$

where J_n denotes the Fejér-Hermite (or Jackson) polynomial of degree $2n$, i.e. (cf. [36II,p.21ff]),

$$(J_n f)(x) := \frac{1}{(2n+1)^2} \sum_{k=0}^{2n} f(x_{kn}) [D_n(x - x_{kn})]^2. \qquad ∎$$

Let us mention that essentially the result of Corollary 3.5 is already contained in [11, p.422], see also [20, p.392ff; 36II, p.18ff].

3.4 Algebraic Lagrange Interpolation at Jacobi Knots

Let $-1<x_{1n}^{\alpha\beta}<\cdots<x_{nn}^{\alpha\beta}<1$ be the zeros of the Jacobi polynomial $(\alpha,\beta>-1)$

$$P_n^{\alpha\beta}(x):=\frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\left(\frac{d}{dx}\right)^n[(1-x)^{\alpha+n}(1+x)^{\beta+n}].$$

Consider the (algebraic) Lagrange polynomial

$$L_n^{\alpha\beta}f:=\sum_{k=1}^{n}f(x_{kn}^{\alpha\beta})l_{kn}^{\alpha\beta}, \qquad l_{kn}^{\alpha\beta}(x):=\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{x-x_{jn}^{\alpha\beta}}{x_{kn}^{\alpha\beta}-x_{jn}^{\alpha\beta}},$$

associated with these knots, f being a continuons function on $[-1,1]$, i.e., $f\in C[-1,1]$ (with usual sup-norm $\|\cdot\|_C$). It is well-known that $L_n^{\alpha\beta}$ is a projection on $\mathscr{P}_{n-1}$, the set of algebraic polynomials of degree at most $n-1$. The operator norm of $L_n^{\alpha\beta}$ is given by

$$\lambda_n^{\alpha\beta}:=\|L_n^{\alpha\beta}\|_{C[-1,1]}=\left\|\sum_{k=1}^{n}|l_{kn}^{\alpha\beta}|\right\|_C$$

which is unbounded by the theorem of Faber. Again we consider the corresponding Fejér-Hermite polynomial

$$(H_n^{\alpha\beta}f)(x):=\sum_{k=1}^{n}f(x_{kn}^{\alpha\beta})h_{kn}^{\alpha\beta}(x),$$

the polynomials $h_{kn}^{\alpha\beta}\in\mathscr{P}_{2n-1}$ satisfing $h_{kn}^{\alpha\beta}(x_{jn}^{\alpha\beta})=\delta_{jk}$ and $(h_{kn}^{\alpha\beta})'(x_{jn}^{\alpha\beta})=0$ (with Kronecker symbol δ_{jk}). It follows that $H_n^{\alpha\beta}$ is a positive operator for $-1<\alpha,\beta\leqslant 0$ (cf. [29,p.339]), and thus $\|H_n^{\alpha\beta}\|_{[C[-1,1]]}=1$ for all $n\in\mathbb{N}$. Hence (2.23) is fulfilled, and Corollary 2.3 delivers (cf. [20,p.389ff]).

Corollary 3.6. *Let $-1<\alpha,\beta\leqslant 0$, and suppose that $\{\varepsilon_n\}$, subject to (2.1), and $\{\psi_n\}$ satisfy (2.24). The following assertions are equivalent:*

(i) $\lambda_n^{\alpha\beta}\varepsilon_n=o(\psi_n)$ $\qquad (\cdots=O(\psi_n))$

(ii) $\|L_n^{\alpha\beta}f-f\|_C=o_f(\psi_n)$ $\qquad (\cdots=O_f(\psi))$

for each $f\in C[-1,1]$ with $E(f,\mathscr{P}_{n-1})=O_f(\varepsilon_n)$.

Note that, e.g., in the Tchebycheff case $\alpha=\beta=-1/2$ one has $\lambda_n^{-1/2,-1/2}\sim\log n$, whereas in the Legendre case $\alpha=\beta=0$ the operator norms $\lambda_n^{0,0}$ behave like $n^{1/2}$ (cf. [29,p.335ff]).

Concerning the error of the Lagrange interpolation at fixed point $x_0\in(-1,1)$, set

$$\lambda_n^{\alpha\beta}(x_0):=\sum_{k=1}^{n}|l_{kn}^{\alpha\beta}(x_0)|.$$

In [29,p.336f] it is shown that $(n\to\infty)$

(3.9) $$\lambda_n^{\alpha\beta}(x_0)=O(\log n),\quad \lambda_n^{\alpha\beta}(x_0)\neq o(\log n).$$

Hence Theorem 2.1 delivers for all $\alpha,\beta>-1$:

Corollary 3.7. *Let $\alpha,\beta>-1$ and $x_0\in(-1,1)$ be fixed. Then for each ω satisfying* (1.5), (2.2) *the condition* $(n\to\infty)$

(i) $$\lambda_n^{\alpha\beta}(x_0)\omega(1/n)=o(1)$$

is sufficient for

(ii) $$|(L_n^{\alpha\beta}f)(x_0)-f(x_0)|=o_f(1)$$

for each $f\in C[-1,1]$ with $\omega_1(t,f;C[-1,1])=o_f(\omega(t))$.

On the other hand, (ii) *necessarily implies*

(iii) $$\liminf_{n\to\infty}\lambda_n^{\alpha\beta}(x_0)\omega(1/n)=0.$$

Proof (i)⇒(ii) follows by the estimate (cf. [20, p. 112, 389f])

(3.10) $$|(L_n^{\alpha\beta}f)(x_0)-f(x_0)|\leqslant C\lambda_n^{\alpha\beta}(x_0)\omega_1(1/n,f;C[-1,1]),$$

where the modulus of continuity of $f\in C[-1,1]$ is defined by

$$\omega_1(t,f;C[-1,1]):=\sup\{|f(x)-f(y)|;-1\leqslant x,y\leqslant 1,|x-y|\leqslant t\}.$$

For the necessity of (iii) set

$$X=C[-1,1],Y=\mathbf{C},U_nf=(L_n^{\alpha\beta}f)(x_0),Uf=f(x_0),$$
$$T_nf=\omega_1(d_n,f;C[-1,1]),d_n:=\min_{1\leqslant k\leqslant n-1}(x_{k+1,n}^{\alpha\beta}-x_{kn}^{\alpha\beta}).$$

In order to construct elements g_n, choose $f_n\in X$ such that

$$\|f_n\|_X=1,|U_nf_n|\geqslant\frac{1}{2}\|U_n\|_{[X,Y]}=\frac{1}{2}\lambda_n^{\alpha\beta}(x_0),$$

and let h be an infinitely often differentiable function on $\mathbb{R}$ with (cf. [5; 6, p. 87ff])

(3.11) $$0\leqslant h(u)\leqslant 1,\qquad h(u)=\begin{cases}0, & u\leqslant 0\\ 1, & u\geqslant 1.\end{cases}$$

Setting

$$g_n(u):=\begin{cases}f_n(x_{jn}^{\alpha\beta})+[f(x_{j+1,n}^{\alpha\beta})-f_n(x_{jn}^{\alpha\beta})]h\left(\dfrac{u-x_{jn}^{\alpha\beta}}{d_n}\right) & \\ \qquad\qquad \text{for } x_{jn}^{\alpha\beta}\leqslant u\leqslant x_{j+1,n}^{\alpha\beta}, 1\leqslant j\leqslant n-1\\ f_n(x_{1n}^{\alpha\beta}) & \text{for } -1\leqslant u\leqslant x_{1n}^{\alpha\beta}\\ f_n(x_{nn}^{\alpha\beta}) & \text{for } x_{nn}^{\alpha\beta}\leqslant u\leqslant 1,\end{cases}$$

$g_n\in C[-1,1]$ is infinitely often differentiable, satisfying

$$\|g_n\|_C\leqslant 1,\|g_n'\|_C\leqslant\|h'\|_C/d_n.$$

Since $U_ng_n=U_nf_n$, conditions (2.4—6) follow with $\varphi_n=d_n$, and thus (2.8) (note that (2.3) is not needed), i.e.,

$$\lambda_n^{\alpha\beta}(x_0)\omega(d_n)=o(1).$$

Moreover, in view of (3.9) and $d_n\sim n^{-2}$ (cf. [29, p.238]) one has that for infinitely many n

$$\omega(n^{-2})\lambda_n^{\alpha\beta}(x_0)\leqslant C\omega(d_n)\log n^2=2C\omega(d_n)\log n\leqslant C^*\omega(d_n)\lambda_n^{\alpha\beta}(x_0).$$

This completes the proof of (iii).

With the aid of quantitative condensation principles as given in [7] one may establish equivalence assertions not only for a fixed $x_0\in(-1,1)$ but also for all x in a dense set of second category in $[-1,1]$ (for relevant results in a concrete setting, but with much stronger negative assertions than those available via the present methods, see e.g. [25; 35]). However, let us also point out that the present analysis is not suitable for treating divergence phenomena on sets of full measure, that is, divergence (almost) everywhere.

At this stage some general historical comments may be in order. In fact, Lagrange interpolation is one of those areas where a rich variety of negative results is known. In this connection the material of Section 3.3, 4 just indicates how to recover (and extend) results of a uniform or pointwise (i. e., at a fixed, prescribed point) nature via the unified approach of Section 2. Apart from the literature already cited, let us mention work of Erdös-Turan (cf. [10]) which was followed up in a number of papers of Kis, Szabados, Vértesi (cf. [12; 13; 34]) and of Privalov (cf. [24; 26]). Note that this collection again reflects our particular interest in those negative results which can be completed to an equivalence assertion via a suitable direct estimate. In fact, the present paper, which actually originates from the material of Section 3.6, could also be considered as a (extended) realization of assertions envisaged by Losinskii in a number of Doklady notes (cf. [15-17]) which appeared 1948-1953 without any indication of proofs (see also the relevant comments in the Mathematical Reviews).

3.5 Interpolatory Quadrature Procedures at Jacobi Knots

For $\alpha,\beta>-1$ and $f\in C[-1,1]$ consider the quadrature formula

$$Q_n^{\alpha\beta}f:=\sum_{k=1}^{n}f(x_{kn}^{\alpha\beta})A_{kn}^{\alpha\beta},\quad A_{kn}^{\alpha\beta}:=\int_{-1}^{1}l_{kn}^{\alpha\beta}(u)\,du,$$

for the approximate calculation of $Qf:=\int_{-1}^{1}f(u)du$. According to [29, p. 355ff], the sequence of operator norms

$$q_n^{\alpha\beta}:=\|Q_n^{\alpha\beta}\|_{[C[-1,1],C]}=\sum_{k=1}^{n}|A_{kn}^{\alpha\beta}|$$

is bounded for $-1<\alpha,\beta\leqslant 3/2$. In fact, the proof given there shows that $q_n^{\alpha\beta}$ behaves asymptotically like $n^{\gamma-3/2}$ if $\gamma:=\max\{\alpha,\beta\}>3/2$.

Corollary 3.8. *Let $\alpha,\beta>-1$ and $\gamma:=\max\{\alpha,\beta\}>3/2$. For each ω satisfying* (1.5), (2.2) $\langle$(1.5)$\rangle$ *and each positive function ψ on $(0,\infty)$ with* (3.6) *the following assertions are equivalent* $(n\to\infty, t\to 0+)$:

(i) $q_n^{\alpha\beta}\omega(1/n)=o(\psi(n))$ $\langle\cdots=O(\psi(n))\rangle$,

(ii) $|Q_n^{\alpha\beta}f-Qf|=o_f(\psi(n))$ $\langle\cdots=O_f(\psi(n))\rangle$

for each $f\in C[-1,1]$ with $\omega_1(t,f;C[-1,1])=O_f(\omega(t))$.

Proof Let h denote an infinitely often differentiable function on R satisfying

(3.11). For fixed $0<r<1$ let $n\in\mathbf{N}$ be such that $|x_{1n}^{\alpha\beta}|, |x_{nn}^{\alpha\beta}|>r$, and set (cf. [33])

$$g_n(u):=\begin{cases} s_{jn}+(s_{j+1,n}-s_{jn})h\left(\dfrac{u-x_{jn}^{\alpha\beta}}{\Delta_n}\right) & \\ \qquad\qquad \text{for } x_{jn}^{\alpha\beta}\leqslant u\leqslant x_{j+1,n}^{\alpha\beta},\ 1\leqslant j\leqslant n-1 \\ 0 \qquad\qquad \text{for } -1\leqslant u\leqslant x_{1n}^{\alpha\beta},\ x_{nn}^{\alpha\beta}\leqslant u\leqslant 1, \end{cases}$$

$$\Delta_n:=\min_{|x_{jn}^{\alpha\beta}|<r}\{x_{jn}^{\alpha\beta}-x_{j-1,n}^{\alpha\beta}, x_{j+1,n}^{\alpha\beta}-x_{jn}^{\alpha\beta}\},$$

$$s_{jn}:=\begin{cases}\operatorname{sgn}A_{jn}^{\alpha\beta}, & |x_{jn}^{\alpha\beta}|\leqslant r\\ 0, & \text{elsewhere.}\end{cases}$$

Then g_n is infinitely often differentiable with

$$\|g_n\|_C\leqslant 1,\quad \|g_n'\|_C\leqslant 2\|h'\|_C/\Delta_n\leqslant Cn, \tag{3.12}$$

since $\Delta_n\sim 1/n$ (cf. [29, p. 238]). Now apply Theorem 2.1 ⟨2.2⟩ to

$$X=C[-1,1],\ Y=\mathbf{C},\ U_n=Q_n^{\alpha\beta},\ U=Q,$$
$$T_nf=\omega_1(1/n,f;C[-1,1]).$$

Since $Q_n^{\alpha\beta}p=Qp$ for each $P\in\mathscr{P}_{n-1}$, thus (2.20), condition (2.3) is a consequence of (2.22) and the classical Jackson theorem (cf. (3.10)), i. e.,

$$|U_nf-Uf|\leqslant Cq_n^{\alpha\beta}E(f,\mathscr{P}_{n-1})\leqslant Cq_n^{\alpha\beta}|T_nf|,$$

whereas (2.4,5) with $\varphi_n=1/n$ follow by (3.12) (cf. (3.3)). Concerning (2.6), one has (cf. [29, p. 358f])

$$q_n^{\alpha\beta}\sim\sum_{|x_{kn}^{\alpha\beta}|<r}|A_{kn}^{\alpha\beta}|=Q_n^{\alpha\beta}g_n. \qquad \blacksquare$$

For material related to Corollary 3.8 see [38; 33].

3.6 Multipliers of Strong Convergence

For a Banach space X let $\{p_k\}_{k\in P}\subset[X]$ ($\mathbf{P}:=$ set of nonnegative integers) be a total sequence of mutually orthogonal projections, i. e., $p_jp_k=\delta_{jk}p_k$ for all $j,k\in\boldsymbol{P}$, and $p_kf=0$ for all $k\in\mathbf{P}$ implies $f=0$. Moreover, assume that this sequence is complete and regular, i. e., the set of polynomials

$$\Pi:=\bigcup_{n=0}^{\infty}\Pi_n,\qquad \Pi_n:=\operatorname{span}\left(\bigcup_{k=0}^{n}p_k(x)\right),$$

is dense in X, and there exists some $\alpha\geqslant 0$ such that the sequence of the operator norms of the Cesàro-(C,α)-means

$$(C,\alpha)_nf:=\sum_{k=0}^{n}(A_{n-k}^{\alpha}/A_n^{\alpha})p_kf,\quad A_n^{\alpha}:=\binom{n+\alpha}{n},$$

is bounded. Let λ be an infinitely often differentiable function satisfying (3.7). Then for the de la Vallée Poussin means $V_nf:=\Sigma_{k=0}^{2n}\lambda(k/n)\ p_kf$ one has (cf. [21, 22])

$$\|V_n\|_{[X]}\leqslant C,\ V_nf\in\Pi_{2n},\qquad V_np_n=p_n\quad\text{for } p_n\in\Pi_n. \tag{3.13}$$

A sequence $\tau:=\{\tau_k\}_{k\in P}\subset\boldsymbol{C}$ is called a multiplier, thus $\tau\in M(X)$, if for each $f\in X$ there exists $f^\tau\in X$ such that $p_kf^\tau=\tau_kp_kf$ for all $k\in\boldsymbol{P}$. Since f^τ is uniquely determined by f, the operator T^τ given by $T^\tau f := f^\tau$ is well-defined, in fact belongs to $[X]$. Moreover, $M(X)$ becomes a Banach algebra under the norm $\|\tau\|_M := \|T^\tau\|_{[X]}$. In these terms, $\tau\in M(X)$ is called a multiplier of strong convergence for a subset $A\subset X$ if for each $f\in A$

$$\lim_{n\to\infty}\|T^{\tau(n)}f-T^\tau f\|_X=0,$$

where the truncated kernel seqnence $\tau(n)$ is defined for each $n\in \boldsymbol{P}$ by

$$\tau(n)_k:=\begin{cases}\tau_k, & o\leqslant k\leqslant n\\ 0, & k>n.\end{cases}$$

Corollary 3.9. *Let $\{\varepsilon_n\}$, subject to (2.1), and $\{\psi_n\}$ satisfy (2.24). The following assertions are equivalent for $\tau\in M(X)$:*

(i) $\|\tau(n)\|_M\varepsilon_n=o(\psi_n)$ $\quad(\cdots=O(\psi_n))$,

(ii) $\|T^{\tau(n)}f-T^\tau f\|_X=o_f(\psi_n)$ $\quad(\cdots=O_f(\psi_n))$

for each $f\in X$ with $E(f,\Pi_n)=O_f(\varepsilon_n)$.

Proof Let ns check the conditions of Corollary 2.3 for $X=Y$, $U_n=T^{\tau(n)}$, $U=T^\tau$. Obviously,

$$U_nV_n=U_n,\quad U_nP_n=UP_n\quad\text{for}\quad P_n\in\Pi_n,$$

thus (2.20,23) (see (3.13)). Concerning (2.21), let $f\in X$ with $\|f\|_X=1$ be snch that $\|\tau\|_M\leqslant 2\|T^\tau f\|_X$. Since Π is dense in X, there exists $p_n\in\Pi_n$ with $\|f-p_n\|_X\leqslant 1/4$ so that for $m\geqslant n$

$$\|\tau\|_M\leqslant 2\|\tau\|_M\|f-p_n\|_X+2\|\tau(m)\|_M\|p_n\|_X,$$

thus $\|\tau\|_M\leqslant 5\|\tau(m)\|_M$. ∎

Let us consider the particular case of one-dimensional trigonometric expansions in $X=C_{2\pi}$, thus ($k\in\boldsymbol{N}$, cf. (1.1))

$$(p_0f)(x)=f^\wedge(0),\quad 2(p_kf)(x)=f^\wedge(k)e^{ikx}+f^\wedge(-k)e^{-ikx}.$$

Obviously, this system is mutually orthogonal, total, fundamental, and regular (take, e. g., $\alpha=1$ and Fejér's theorem) in $C_{2\pi}$. Therefore, since for $\tau_k=1$, $k\in\boldsymbol{P}$, the operator $T^{\tau(n)}$ is the partial sum operator (1.2), Corollary 3.9 first of all reestablishes Corollary 3.2. Moreover,

$$\|\tau(n)\|_{M(C_{2\pi})}=\|D_n^\tau\|_1:=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=-n}^{n}\tau_{|k|}e^{iku}\right|du$$

so that condition (i) of Corollary 3.9 takes on the form $\|D_n^\tau\|_1\varepsilon_n=o(\psi_n)$. Of course, the characterization of smoothness in Corollary 3.9 (ii) via the functional of best approximation can now be replaced by ordinary Lipschitz conditions if one applies Theorem 2.1 directly to the present situation with $T_nf=\omega_1(1/n,f;C_{2\pi})$. In fact, for the verification of the conditions one may proceed as for Corollary 3.1 (cf. (3.2, 3)) to obtain for $\psi_n=1$, $n\in\boldsymbol{N}$:

Corollary 3.10. *For ω satisfying (1.5), (2.2) the condition $\|D_n^\tau\|_1\omega(1/n)=o(1)$*

is necessary and sufficient for $\tau \in M(C_{2\pi})$ *to be a multiplier of uniform convergence for the Lipschitz class* $\omega_1(t,f;C_{2\pi}) = O_f(\omega(t))$.

If, in an obvious terminology, one applies Theorem 2.1 to $X = L^1_{2\pi}$, $Y = C_{2\pi}$, then in view of $\|\tau(n)\|_{M(L^1_{2\pi}, C_{2\pi})} = \|D^\tau_n\|_C$

Corollary 3.11. *For* ω *satisfying* (1.5), (2.2) *the condition* $\|D^\tau_n\|_C\, \omega(1/n) = o(1)$ *is necessary and sufficient for* $\tau \in M(L^1_{2\pi}, C_{2\pi})$ *to be a multiplier of uniform convergence for the Lipschitz class* $\omega_1(t,f;L^1_{2\pi}) = O_f(\omega(t))$.

Corollary 3.10 and 3.11 are due to Teljakovskii [30] and Pochuev[23], respectively. As already mentioned in Section 3.4, it was their work which mainly influenced the development of the quantitative uniform boundedness principles as outlined in Section 2. In fact, the results of Teljakovskii and Pochuev were first extended in [18;19] from the particular situation of 2π-periodic functions and one-dimensional trigonometric expansions to regular biorthogonal systems in Banach spaces (cf. Corollary 3.9). This in turn paved the way for the general approach given in[4-6;9] whose main features were reproduced in Section 2 and which in particular does not need any orthogonal structure or projection properties.

References

[1] Butzer, P. L., Some recent applications of functional analysis to approximation theory. In: Euler Festschrift, Berlin 1983/4 (in print).

[2] Butzer, P. L., Berens, H., Semi Groups of Operators and Approximation. Springer, Berlin 1967.

[3] Butzer, P. L., Nessel, R. J., Fourier Analysis and Approximation, Vol. I. Birkhäuser, Basel, and Acad. Press, New York 1971.

[4] Dickmeis, W., Nessel, R. J., A unified approach to certain counterexamples in approximation theory in connection with a uniform boundedness principle with rates. J. *Approx. Theory* 31 (1981), 161—174.

[5] Dickmeis, w., Nessel, R. J., On uniform boundedness principles and Banach-Steinhaus theorems with rates. Numer. *Funct. Anal. Optim.* 3 (1981), 19—52.

[6] Dickmeis, W., Nessel: R. J., Quantitative Prinzipien gleichmäßiger Beschränktheit und Schärfe von Fehlerabschätzungen. Forschungsberichte des Landes NRW, 3117, Westdeutscher Verl., Opladen 1982.

[7] Dickmeis, W., Nessel, R. J., Condensation principles with rates. *Studia Math.* 75 (1982), 55—68.

[8] Dickmeis, W., Nessel, R. J., van Wickeren, E., Steckin type estimates for locally divisible multipliers in Banach spaces. (in print).

[9] Dickmeis, W., Nessel, R. J., van Wickeren, E., A general approach to counterexamples in numerical analysis. (in print).

[10] Erdös, P., Turán, P., On the role of the Lebesgue functions in the theory of the Lagrange interpolation. *Acta Math. Acad. Sci.* Hungar. 6 (1955), 47—65.

[11] Faber, G., Über stetige Funktionen (zweite Abhandlung). *Math. Ann.* 69(1910), 372—443.

[12] Kis, O., A remark on the order of the error in interpolation (Russian). *Acta Math. Acad. Sci* Hungar. 20 (1969), 339—346.

[13] Kis, O., Szabados, J., Notes on the convergence of the Lagrange interpolation (Russian). *Acta Math. Acad. Sci.* Hungar. 16 (1965), 389—430.

[14] Lebesgue, H., Sur la représentation trigonométrique approchée des fonctions satisfaisant à une condition de Lipschitz. *Bull. Soc. Math.* France 38 (1910), 184—210.

[15] Losinskii, S. M., The spaces $\tilde{C}_\omega$ and $\tilde{C}^*_\omega$ and the convergence of interpolation processes in

them (Russian). *Dokl Akad.* Nauk SSSR 59 (1948), 1389—1392; *MR* 10 (1949), 188.

[16] Losinskii, S. M., On the divergence at a fixed point of interpolation processes (Russian). *Dokl. Akad.* Nauk SSSR 72 (1950), 1017—1020; *MR* 13 (1952), 118.

[17] Losinskii, S. M., On the rapidity of convergence of a sequence of linear operations(Russian). Dokl. *Akad. Nauk SSSR* 89 (1953), 609—612; *MR* 15 (1954), 136.

[18] Mertens, H. J., Nessel, R. J., An equivalence theorem concerning multipliers of strong convergence. *J. Approx. Theory* 30 (1980), 284—308.

[19] Mertens, H. J., Nessel, R. J., Quasikonvexe Multiplikatoren starker konvergenz. *Anal. Math.* 7 (1981), 49—67.

[20] Natanson, I. P., Konstruktive Funktionentheorie. Akad. Verl., Berlin 1955.

[21] Nessel, R. J., Wilmes, G., Nikolskii-type inequalities in connection with regular spectral measures. *Acta Math. Acad. Sci.* Hungar. 33 (1979), 169—182.

[22] Nessel, R. J., Wilmes, G., Über Ungleichungen vom Bernstein-Nikolskii-Riesz-Type in Banach Räumen. Forschungsberichte des Landes NRW, 2841, Westdeutscher Verl., Opladen 1979.

[23] Pochuev, V. R., On multipliers of uniform convergence and multipliers of uniform boundedness of partial sums of Fourier series (Russian). *Izv. Vyss. Ucebn. Zaved. Matematika* 21 (1977), 74—81 = *Soviet Math.* 21 (1977), 60—66.

[24] Privalov, A. A., Divrgence of interpolation processes at a fixed point (Russian). *Mat. Sb.* 66. (1965), 272—286.

[25] Privalov, A. A., Divergence of interpolation processes on sets of the second category (Russian). *Mat. Zametki* 16 (1975), 175—183 = *Math*, Notes 18(1675), 692—694.

[26] Privalov, A. A. On linear polynomial operators and their convergence. In: Approximation and Function Spaces, Proc. Conf. Gdańsk 1979 (Z. Ciesielski, Ed.), North-Holland 1981, 584—603.

[27] Stein, E. M., Weiss, G., Introduction to Fourier Analysis on Euclidean Spaces. Princton Univ. Press, Princeton (N. J.)1971.

[28] Szabados, J., On the convergence of the interpolatory quadrature procedures in certain classes of functions. *Acta Math. Acad. Sci.* Hungar. 18 (1967), 97—111.

[29] Szegö, G., Orthogonal Polynomials (*4th* edition). *Amer. Math. Soc. Celloq. Publ.* 23, Providence (R. I.)1975.

[30] Teljakovskii, S. A., Uniform convergence factors for Fourier series of functions with a given modulus of continuity (Russian). *Mat. Zametki* 10 (1971), 33—40 = *Math. Notes* 10(1971), 444—448.

[31] Timan, A. F., Theory of Approximation of Functions of a Real Variable. Pergamon Press, New York 1963.

[32] Vértesi, P. O. H., On certain linear operators. III (On the divergence of Fourier series). *Acta Math. Acad. Sci.* Hungar. 23(1972), 109—113.

[33] Vértesi, P. O. H., On certain linear operators. VII (A summary from new point of view. Estimations for mechanical quadratures). *Acta Math. Acad. Sci.* Hungar. 25(1974), 67—80.

[34] Vértesi, P. O. H., On certain linear operators. VIII (Functions where $\omega_m(f,t)=O[\omega_m(t)]$ including $\omega_m(t)=t^m$ also. Functions having bounded derivatives). *Acta Math. Acad. Sci.* Hungar. 25 (1974), 171—187.

[35] Vértesi, P. O. H., Divergence of Lagrange interpolation on a set of second category.(in print).

[36] Zygmund, A., Trigonometric Series I, II. Cambridge Univ. Press, Cambridge(Mass.)1959.