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Some Theorems of Analytic Functions*

Hu Ke

(Jiangxi Teacher's University)

1 Introduction

We denote that: σ— the class of functions $\omega(z)=A_1z+A_2z^2+\cdots$ regular in the unit disk such that $\sum_{n=1}^{\infty}n|A_n|^2<\infty$; K_c— the class of close-to-convex function $f(z)$, that is, if $f(z)=a_1z+a_2z^2+\cdots$ there exists a starlike function $g(z)=b_1z+b_2z^2+\cdots$ such that

$$\operatorname{Re}\left\{z\frac{f'(z)}{g(z)}\right\}>0.$$

Milin[1] proved that:

Theorem If the Taylor coefficients of a function $\omega(z)\in\sigma$ satisfy the condition $\operatorname{Re}\left\{\sum_{k=1}^{n}A_k\right\}=O(1)\ (n\to\infty)$, then for any $h>\frac{1}{2}$ the function $\varphi(z)=e^{\omega(z)}=\sum_{k=0}^{\infty}D_kz^k$, satisfies the asymptotic equality

$$Q_n(h)=\frac{\{\varphi(z)\,(1-z)^{-h}\}}{d_n(h)}\sim\varphi(r)\sim\exp\left\{\sum_{k=1}^{n}A_k\right\}\quad(n\to\infty),\tag{1}$$

where $r=1-\frac{1}{n}$, $d_n(h)$ defined *by* $\frac{1}{(1-x)^h}=\sum_{n=0}^{\infty}d_n(h)x^n$,

The case $0<h\leqslant\frac{1}{2}$ is open.

In this paper we have the following theorems.

Theorem 1 Let $\omega(z)\in\sigma$ such that (i) $\operatorname{Re}\left\{\sum_{k=1}^{n}A_k\right\}=O(1)$;

(ii) $Q_n(h)-Q_m(h)\to0$ as $n\to\infty$ and $\frac{n}{m}\to1$. Then for any $h>0$, the asymptotic equality (1) holds.

Theorem 2 Let $f(z)=z+a_2z^2+\cdots\in K_c$ and $\psi(z)=\left\{\frac{f(z)}{z}\right\}^{\lambda}=1+\sum_{n=1}^{\infty}D_n^{(\lambda)}z^n$. If $\lim_{\rho\to1}\frac{(1-\rho)^2}{\rho}|f(\rho e^{i\rho_0})|=a\neq0$, then for $\frac{1}{2}<\lambda\leqslant1$,

$$\big||D_n(\lambda)|-|D_{n-1}(\lambda)|\big|\leqslant A\,n^{2\lambda-2},\quad n=2,3\cdots\tag{2}$$

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where constant A only depends on a. Moreover, for $\lambda > \frac{1}{2}$,

$$\frac{D_n(\lambda) - D_{n-1}(\lambda) e^{-i\varphi_0}}{d_n(2\lambda - 1)} \sim a^\lambda \exp\{i[\arg f(re^{i\varphi_0}) - (n+\lambda)\varphi_0]\}, \tag{3}$$

as $n \to \infty \left(r = 1 - \frac{1}{n}\right)$.

2 Proof of Theorem 1

Lemma 1[2] If $b_n \geqslant 0$, $g(x) = \sum_{n=0}^{\infty} b_n x^n \sim C(1-x)^{-a}$, $(x \to 1)$ and $a > 0$, then

$$\sum_{k=1}^{n} \frac{b_k}{d_k(a)} \sim \frac{Cn}{\Gamma(a)}, \tag{4}$$

where C is an absolute constant.

Lemma 2 If $\sum_{n=1}^{\infty} n|A_n|^2 < \infty$, then

$$\sum_{k=1}^{n} \frac{\sum_{l=1}^{k} d_{k-l}(h)\, l^2 |A_l|^2}{d_k(h+1)} = o(n) \quad (n \to \infty). \tag{5}$$

To prove (5), we note that

$$\sum_{n=1}^{\infty} n^2 |A_n|^2 x^n = o\left(\frac{1}{1-x}\right),$$

implies

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} d_{n-k}(h)\, |kA_k|^2\right) x^n = o\left(\frac{1}{(1-x)^{h+1}}\right). \tag{6}$$

Applying Lemma 1 to (6), (5) follows at once.

To derive Theorem 1, we begin with an identity

$$\left\{\frac{\varphi(z)(1-z)^{-h}}{d_n(h)}\right\}_n - \left\{\frac{\varphi(z)(1-z)^{-h-1}}{d_n(h+1)}\right\}_n = \frac{\sum_{k=1}^{n} S_{n-k}^{(h)} k A_k}{(n+h) d_n(h)} = B_n, \tag{7}$$

where $S_l^{(h)} = \sum_{k=0}^{l} d_{l-k}(h) D_k$.

Next applying Cauchy's inequality to B, we get

$$\sum_{l=1}^{n} |B_l| \leqslant \sum_{l=1}^{n} \sqrt{\frac{\sum_{k=1}^{l} \frac{|B_k^{(h)}|^2}{d_k(h)}}{d_l(h+1)} \sum_{k=1}^{l} \frac{d_{l-k}(h) |kA_k|^2}{d_l(h+1)}}$$

$$\leqslant \sqrt{\sum_{l=1}^{n} \frac{\sum_{k=1}^{l} \frac{|S_k^{(h)}|^2}{d_k(h)}}{d_l(h+1)} \sum_{l=1}^{n} \sum_{k=1}^{l} \frac{d_{l-k}(h) |kA_k|^2}{d_l(h+1)}}. \tag{8}$$

In [3] we proved that

$$\sum_{k=1}^{l} \frac{|S_k^{(h)}|^2}{d_k(h)} = O(d_l(h+1)) \tag{9}$$

and this, together with Lemma 2 and (8) yield the result

$$\sum_{l=1}^{n}|B_l| = o(n). \tag{10}$$

Milin Theorem says that $Q_n(h+1)$ converges to a fixed number. Hence $Q_n(h+1)-Q_m(h+1)\to 0$ as $n\to\infty$, $\frac{n}{m}\to 1$. By the assumption of Theorem we have

$$||B_n|-|B_m||\leqslant|Q_n(h)-Q_m(h)|+|Q_n(h+1)-Q_m(h+1)|=o(1).$$

From (10) and Schmidt-Tauber Theorem we have $B_n\to 0$ $(n\to\infty)$. i. e.,

$$[Q_n(h)-Q_n(h+1)]\to 0,\quad (n\to\infty). \tag{11}$$

The asymptotic equality (10) and above Milin Theorem in combination yield the required assertion.

3 Proof of Theorem 2

It is easy to see that: (i) in virtue of the known inequality $\left|\frac{f(z)}{z}\right|^2(1-|z|^2)\leqslant|f'(z)|$ and $\left|\frac{b_1zf'(z)}{g(z)}\right|\leqslant\frac{1+|z|}{1-|z|}$. The assumption gives

$$a^2\leqslant\frac{(1-r)^4}{r^2}|f(re^{i\varphi_0})|^2\leqslant\frac{(1-r)^3}{1+r}\left|\frac{f'(re^{i\varphi_0})}{g(re^{i\varphi_0})}b_1r\right|\left|\frac{g(re^{i\varphi_0})}{b_1r}\right|$$

$$\leqslant\frac{(1-r)^2}{r}|g(re^{i\varphi_0})|\,|b_1|^{-1}.$$

This shows that φ_0 is also a direction of Maximum growth of the function $g(z)$.

Let $\log\frac{f(z)}{z}=2\sum_{n=1}^{\infty}\alpha_nz^n$ and $\log\frac{g(z)}{b_1z}=2\sum_{n=1}^{\infty}\beta_nz^n$. Then by Bazilevic Theorem, we have

$$\sum_{k=1}^{\infty}k|\alpha_k-\beta_k|^2\leqslant 2\sum_{k=1}^{\infty}k\left|\alpha_k-\frac{e^{-ik\varphi_0}}{k}\right|^2+2\sum_{k=1}^{\infty}k\left|\alpha_k-\frac{e^{-ik\varphi_0}}{k}\right|^2<3\log\frac{1}{a}.$$

The inequality has been proved by Pan Yi-Fei.

(ii) Since $\mathrm{Re}\,p(z)=\mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right)>0$, the integral representation of $p(z)$ is

$$P(z)=\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}dr(t)+i\,\mathrm{In}\left(\frac{1}{b_1}\right), \tag{12}$$

where $r(t)$ is an increasing function and $r(2\pi)-r(0)=\mathrm{Re}\left(\frac{1}{b_1}\right)$. Hence, for $\alpha>1$,

$$\int_0^{2\pi}|P_1(re^{i\theta})|^{\alpha}d\theta\leqslant\int_0^{2\pi}\left|\int_0^{2\pi}\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}dr(t)\right|^{\alpha}d\theta$$

$$\leqslant 2^{\alpha}\int_0^{2\pi}\int_0^{2\pi}\frac{dr(t)d\theta}{|e^{it}-re^{i\theta}|^{\alpha}}\leqslant\frac{A}{(1-r)^{\alpha-1}}, \tag{13}$$

because $\int_0^{2\pi}|1-re^{i\theta}|^{-\alpha}d\theta\leqslant A(1-r)^{1-\alpha}$ for $\alpha>1$. Then we consider the obvious identity

$$(1-z)\left\{\frac{f(z)}{z}\right\}^{\lambda}=\frac{1}{(1-z)^{2\lambda-1}}\left\{\frac{f(z)}{z}(1-z)^2\right\}^{\lambda}=\varphi(z)(1-z)^{-h},\ h=2\lambda-1$$

and denoting

$$\omega(z)=\log\left\{\frac{f(z)}{z}(1-z)^2\right\}^{\lambda}=\sum_{k=1}^{\infty}A_kz^k,\qquad A_k=2\lambda\left(\alpha_k-\frac{1}{k}\right),$$

$$\varphi(z)=e^{\omega(z)}=\sum_{k=0}^{\infty}D_k^{(\lambda)}z^k,$$

Milin[1] proved that the Taylor coefficients of $\omega(z)$ satisfies the conditions of Theorem 2:

$$\sum_{k=1}^{\infty}k|A_k|^2<\infty,$$

$$\mathrm{Re}\left\{\sum_{k=1}^{n}A_k\right\}=O(1),\ (n\to\infty).$$

Therefore in order to prove (3) it is sufficient to prove $(Q_n-Q_m)\to 0$ as $n\to\infty$, where $Q_k=\{D_k(\lambda)-D_{k-1}(\lambda)\}/d_k(2\lambda-1)$. To do this, we estimate some integrals. We assume that $\varphi_0=0$ since otherwise we need only consider the function $e^{-i\varphi_0}f(e^{i\varphi_0}z)$. Let δ such $(2\lambda-1)>\delta>0$, we observe that $\Phi(z)=\left\{(1-z)^{1+\delta}\left\{\frac{f(z)}{z}\right\}^{\lambda-1}\left\{\frac{g(z)}{b_1z}\right\}\right\}^{\frac{\beta}{2}}$ is also regular and nonvanish in $|z|<|$. Hence by Lebejev-Milin inequality, we have

$$\frac{1}{2\pi}\int_0^{2\pi}|\Phi(re^{i\varphi})|^2d\theta\leqslant\exp\left\{\beta^2\sum_{k=1}^{\infty}k\left|\beta_k-(1-\lambda)\alpha_k-\frac{1+\delta}{2k}\right|^2r^{2k}\right\}.$$

Since,

$$\sum_{k=1}^{\infty}k\left|\beta_k-(1-\lambda)\alpha_k-\frac{1+\delta}{2k}\right|^2r^{2k}=(1-\lambda)\sum_{k=1}^{\infty}k|\beta_k-\alpha_k|^2r^{2k}$$

$$+\lambda\sum_{k=1}^{\infty}k\left|\beta_k-\frac{1}{k}\right|^2r^{2k}-(1-\lambda)\lambda\sum_{k=1}^{\infty}k\left|\alpha_k-\frac{1}{k}\right|^2r^{2k}$$

$$+(2\lambda-1-\delta)\mathrm{Re}\sum_{k=1}^{\infty}\left(\beta_k-\frac{1}{k}\right)r^{2k}(1+\delta-2\lambda)(1-\lambda)\mathrm{Re}\sum_{k=1}^{\infty}\left(\alpha_k-\frac{1}{k}\right)r^{2k}+\left(\frac{2\lambda-1-\delta}{2}\right)^2\log\frac{1}{1-r^2}$$

$$\leqslant 4\log\frac{1}{a}+\left(\frac{2\lambda-1-\delta}{2}\right)^2\log\frac{1}{1-r^2},$$

by (13) and assumption of Theerem 2, i. e.,

$$\frac{1}{2\pi}\int_0^{2\pi}|\Phi(re^{i\theta})|^2d\theta\leqslant\frac{A}{(1-r)^{\frac{(2\lambda-1-\delta)^2\beta^2}{2}}}\tag{14}$$

It is clear that

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1-(re^{i\theta})^k}{1-re^{i\theta}}\right|^2 d\theta=\frac{1-r^{2k}}{1-r^2}. \tag{15}$$

Using the known inequality $\frac{(1-r)^2}{r}|f(re^{i\theta})|\leqslant\frac{(1-r^2)^2}{r^2}|f(r^2e^{i\theta})|$ **and Goluzin's inequality** $|r-r^2e^{i\theta}||f(r^2e^{i\theta})|\leqslant\frac{2r^3}{\sqrt{(1-r^2)^2(1+r^2)}}$, **we have the integral**

$$\int_0^{2\pi}|1-z^k|^{\frac{\delta}{2}}|f|^\lambda d\theta\leqslant\max_{|z|=r}\left(|r-r^2e^{i\theta}||f(re^{i\theta})|\right)^{\frac{\delta}{2}}\int_0^{2\pi}\left|\frac{1-z^k}{1-z}\right|^{\frac{\delta}{2}}|f|^{\lambda-\frac{\delta}{2}}d\theta$$

$$\leqslant\frac{A}{(1-r)^{\frac{\delta}{2}}}\left(\int_0^{2\pi}\left|\frac{1-z^k}{1-z}\right|^2 d\theta\right)^{\frac{\delta}{4}}\left(\int_0^{2\pi}|f|^{\frac{4}{4-\delta}(\lambda-\frac{\delta}{2})}d\theta\right)^{1-\frac{\delta}{4}},\quad (z=re^{i\theta})$$

$$\leqslant A\frac{(1-r^k)^{\frac{\delta}{4}}}{(1-r)^{2\lambda-1}}. \tag{16}$$

The estimate $\int_0^{2\pi}|f(re^{i\theta})|^\alpha d\theta\leqslant A\frac{1}{(1-r)^{2\alpha-1}}$ **is used** $\left(\alpha>\frac{1}{2}\right)$.

Now let $S_n=D_n(\lambda)-D_{n-1}(\lambda)=Q_nd_n(2\lambda-1)$, **we shall show that** $S_n-S_m\to o(d_n(2\lambda-1))$ **as** $n\to\infty$, $n\sim m$. **Indeed, for** $z=re^{i\theta}$, $k=n-m$ $(n>m)$

$$\left|nS_n-mS_m\right|=\frac{1}{2\pi}\left|\int_{|z|=r}\left(\frac{1}{z^n}-\frac{1}{z^m}\right)\frac{\Phi'(z)}{z}dz\right|$$

$$\leqslant\frac{\lambda r^{-n-\lambda-1}}{2\pi}\int_0^{2\pi}\{|1-z^k||1-z||f'||f|^{\lambda-1}+3|1-z^k||f|^\lambda\}d\theta$$

$$\leqslant\frac{\lambda r^{-n-\lambda-1}}{2\pi}\int_0^{2\pi}\left|\frac{1-z^k}{1-z}\right|^\delta|1-z|^{1+\delta}\left|\frac{f'}{g}b_1z\right|\left|\frac{g(z)}{b_1z}\right|d\theta$$

$$+\frac{\lambda r^{-n-\lambda-1}}{z^{\frac{1}{2}\delta}\pi}\int_0^{2\pi}|1-z^k|^{\frac{\delta}{2}}|f|^\lambda d\theta \tag{17}$$

$$=A(I_1+I_2).$$

Applying Holder's inequality to the first integral on the right of (24), we get

$$I_1\leqslant\left(\int_0^{2\pi}\left|\frac{1-z^k}{1-z}\right|^2 d\theta\right)^{\frac{\delta}{2}}\left(\int_0^{2\pi}\left|\frac{b_1zf'}{g}\right|^{\frac{2}{1-2\lambda}}d\theta\right)^{\frac{1}{2}-\lambda}\left(\int_0^{2\pi}|\Phi(z)|^2d\theta\right)^{\frac{1}{\beta}}.$$

where $\frac{1}{\beta}=\lambda-\frac{1}{2}-\frac{\delta}{2}$.

From (13), (14) and (15), we have

$$I_1\leqslant A\frac{(1-r^k)^{\frac{\delta}{2}}}{(1-r)^{2\lambda-1}}. \tag{18}$$

Substituting (15) and (17) into (16), then

$$|nS_n - mS_m| \leqslant \frac{A}{r^{n+\lambda-1}}(n-m)^{\frac{\delta}{4}}(1-r)^{1+\frac{\delta}{4}-2\lambda}. \tag{19}$$

Obviously, if $m=0$, and $r=1-\frac{1}{n}$, then (19) gives

$$|S_n| \leqslant An^{2\lambda-2}.$$

Moreover,

$$n|S_n - S_m| \leqslant (n-m)|S_m| + A\left(\frac{n-m}{n}\right)^{\frac{\delta}{4}} n^{2\lambda-1}.$$

This mesns that $S_n - S_m = o(d_n(2\lambda-1))$ as $n\to\infty$, $n\sim m$, which is the required result. The proof of Theorem 2 is completed.

References

[1] Milin, I, M., Univalent Functions and Orthonormal Systems p. 44—68.
[2] Hu Ke, *Chinese Annals of Matheimatics*, 3 (3) 1982, p. 293—300.
[3] Hu Ke, *Chinese Annals of Matheimatics*, (to appear).