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L^p-orthogonality in Banach Spaces*

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The notion of L^p-orthogonality was posed by F. Sullivan [1], but it had never been studied. The purpose of this paper is to give some results concerning the L^p-orthogonality in Banach spaces.

Throughout this paper, X is a real Banach space,

Definition 1 Let $p>1$ be a fixed real number and let $x,y\in X$. Then x is called L^p-orthogonal to y $(x\perp_{L^p}y)$ iff $\|x+y\|^p=\|x\|^p+\|y\|^p$.

Clearly, L^2-orthogonality is just the Pythagokean orthognality.

For L^p-orthogonality in Banach spaces to have a useful meaning, it is necessary to know that there exist nonzero L^p-orthogonal elements.

Theorem 1 Let $p>1$ be a fixed real number. If $x\neq 0, y\in X$, then there exists a number a such that $x\perp_{L^p}(ax+y)$.

Proof Set $f(t)=\|x+(tx+y)\|^p-\|x\|^p-\|tx+y\|^p$. Clearly, f is a continuous function on $-\infty<t<+\infty$, and we have

$$f(t)=|t|^p[\|x+\frac{1}{t}(x+y)\|^p-\|x\|^p]-|t|^p[\|x+\frac{1}{t}y\|^p-\|x\|^p]-\|x\|^p.$$

Then, for $t\neq 0$,

$$\frac{f(t)}{|t|^{p-1}\operatorname{sgn}t}=\frac{\|x+\frac{1}{t}(x+y)\|^p-\|x\|^p}{\frac{1}{t}}-\frac{\|x+\frac{1}{t}y\|^p-\|x\|^p}{\frac{1}{t}}-\frac{\|x\|^p}{|t|^{p-1}\operatorname{sgn}t}$$

and

$$\lim_{t\to\pm\infty}\frac{f(t)}{|t|^{p-1}\operatorname{sgn}t}=p\|x\|^{p-1}N_\pm(x,x+y)-p\|x\|^{p-1}N_\pm(x,y).$$

Here $N_+(x,y)$ and $N_-(x,y)$ are respectively the right and left Gâteaux derivatives of the norm at x in the direction of y. since $N_\pm(x,x+y)=\|x\|+N_\pm(x,y)$ [2] we have

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$$\lim_{t\to\pm\infty} \frac{f(t)}{|t|^{p-1}\mathrm{sgn}t} = p\|x\|^p > 0.$$

Therefore $f(t)\to+\infty$ as $t\to+\infty$ and $f(t)\to-\infty$ as $t\to-\infty$. Hence there is a number a such that $f(a)=0$, which was to be proved.

An orthogonality $\perp$ is called left (right) unique if for any elements $x\neq 0$, $y\in X$, there exists only one number a such that $ax+y\perp x$ $(x\perp ax+y)$. For L^p-orthogonality, left and right uniqueness are equivalent, since L^p-orthogonality is symmetric.

Theorem 2 Let $p>1$ be a fixed real number. Then L^p-orthogonality in X is unique.

Proof Let us assume that L^p-orthogonality in X is not unique. Then there must exist $x\neq 0$, $z\in X$, and two numbers a_1, a_2 such that $x\perp_{L^p}a_1x+z$ and $x\perp_{L^p}a_2x+z$, without loss of generality, we can assume $a_2>a_1$. Set $y=a_1x+z$ and $a=a_2-a_1$, we have $a>0$ and $x\perp_{L^p}y$ and $x\perp_{L^p}ax+y$.i.e.

$$\|x+y\|^p=\|x\|^p+\|y\|^p \text{ and } \|x+ax+y\|^p=\|x\|^p+\|ax+y\|^p. \tag{1}$$

Setting $g(t)=\|y+tx\|^p$, we have, by (1),

$$g(1)=\|x\|^p+g(0), \tag{2}$$

$$g(a+1)=\|x\|^p+g(a). \tag{3}$$

Clearly g is a convex function on $-\infty<t<+\infty$ [3].

Suppose $0<a<1$, for t_1 and t_2 such that $g(t_1)\neq g(t_2)$, we have

$$g[at_1+(1-a)t_2]=\|a(y+t_1x)+(1-a)(y+t_2x)\|^p\leqq[a\|y+t_1x\|+(1-a)\|y+t_2x\|]^p$$

$$<a\|y+t_1x\|^p+(1-a)\|y+t_2x\|^p=ag(t_1)+(1-a)g(t_2). \tag{4}$$

Therefore, by using (2), (3) and (4) we get

$$g(a)=g[a\cdot 1+(1-a)\cdot 0]<ag(1)+(1-a)g(0), \tag{5}$$

and

$$g(1)=g[a\cdot a+(1-a)(1+a)]<ag(a)+(1-a)g(a+1)$$
$$=ag(a)+(1-a)(g(a)+g(1)-g(0)),$$

and that yields

$$ag(1)+(1-a)g(0)<g(a),$$

contradicting (5).

Now suppose $a>1$, we use convexity of g, and (2) and (3) to obtain $g(0)\neq g(a)$, and $g(1)\neq g(a+1)$, and then by (4) we get

$$g(1)=g\left(\frac{a-1}{a}\cdot 0+\frac{1}{a}\cdot a\right)<\frac{a-1}{a}g(0)+\frac{1}{a}g(a), \tag{6}$$

and

$$g(c) = \left[\frac{1}{a}\cdot 1 + \frac{a-1}{a}(a+1)\right] < \frac{1}{a}g(1) + \frac{a-1}{a}g(a+1)$$

$$= \frac{1}{a}g(1) + \frac{a-1}{a}[g(a) + g(1) - g(0)],$$

contradicting (6).

In the case $a=1$ we have

$$g(2) = g(1) + \|x\|^p = g(0) + 2\|x\|^p,$$

and hence $g(0) \neq g(2)$, and then by (4) we get

$$g(1) = g\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 2\right) < \frac{1}{2}[g(0) + g(2)] = g(0) + \|x\|^p,$$

which is false. Thus in all cases we get a contradiction. Hence L^p-orthogonality is unique in X.

Obviously L^p-orthogonality is not homogeneous or additve in a general Banach space. However, the assumption of these properties will be shown to imply that the space is uniformly convex.

Definition 2 (Clarkson [4]), A Banach space X will be said to be uniformly convex iff to each ε, $0<\varepsilon\leqq 2$, there corresponds a $\delta(\varepsilon)$ such that the conditions

$$\|x\| = \|y\| = 1, \qquad \|x-y\| \geqq \varepsilon$$

imply

$$\left\|\frac{x+y}{2}\right\| \leqq 1 - \delta(\varepsilon).$$

Recall that x is called isosceles orthogonal to y ($x\perp_I y$) iff $\|x+y\| = \|x-y\|$, and if $x\neq 0, y\in X$ then there exists a number b such that $x\perp_I bx+y$, and if $\|y\| \leqq \|x\|$ then $|b| \leqq \frac{\|y\|}{\|x\|}$ (James [5]). We now turn to some sufficient conditions for uniformly convex spaces.

Theorem 3 Let $p>1$ be a fixed real number. If $x,y\in X$ and $\|x\| = \|y\|$ implies $x+y\perp_{L^p} x-y$, then X is a uniformly convex space;

Proof Let $\|x\| = \|y\| = 1$. Then $x+y\perp_{L^p} x-y$, and therefore

$$\|x+y\|^p + \|x-y\|^p = \|(x+y) + (x-y)\|^p = 2^p.$$

Thus if $\|x-y\| \geqq \varepsilon$ $(0<\varepsilon\leqq 2)$, we see that

$$\left\|\frac{x+y}{2}\right\| \leqq \left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^{\frac{1}{p}},$$

so that an admissible value for $\delta(\varepsilon)$ is $1 - \left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^{\frac{1}{p}}$.

Corollary 4 Let $p>1$ be a fixed real rumber. If isosceles orthogonality implies L^p-orthogonality in X, then X is a uniformly convex space.

It is clear since if $x,y\in X$ then $\|x\|=\|y\|$ implies $x+y\perp_I x-y$, and so $x+y\perp_{L^p}x-y$.

Lemma 5 Let $p>1$ be a fixed real rumber, and x, $y\in X$. Then the following are equivalent:

(1) $x\perp_{L^p}y$ implies $x\perp_I y$;

(2) $x\perp_I y$ implies $x\perp_{L^p}y$.

Proof To prove $(1)\Rightarrow(2)$ let us first show that if (1) holds, then X is strictly convex (X is called strictly convex if $0<k<1$, $x\neq y\in X$ imply $\|kx+(1-k)y\|<k\|x\|+(1-k)\|y\|$). If not then there exist $x\neq y\in X$ such that $\|x\|=\|y\|=\left\|\frac{x+y}{2}\right\|=1$, and it follows $x\not\perp_{L^p}y$. By Theorem 1 there exists a number $a\neq 0$ such that $x\perp_{L^p}ax+y$, i.e.,

$$\|x+ax+y\|^p=\|x\|^p+\|ax+y\|^p=1+\|ax+y\|^p. \tag{7}$$

Therefore $x\perp_I ax+y$, and since $\|x\|=\|y\|$, $|a|\leqq\frac{\|y\|}{\|x\|}=1$. From equation (7) we get

$$(2+a)^p=(2+a)^p\left\|\frac{(1+a)x+y}{2+a}\right\|^p=1+\|ax+y\|^p\geqq 1.$$

That means $a\geqq -1$. Thus $a=-1$. Then, from (7)

$$1=\|y\|^p=\|x-y\|^p+1,$$

which contradicts the assumption that $x\neq y$. Thus X is strictly convex. Now suppose that (1) does not imply (2), Then there exist $x,y\in X$ such that $x\perp_I y$ but $x\not\perp_{L^p}y$. By Theorem 1 we can choose $a\neq 0$ such that $x\perp_{L^p}ax+y$. But then, by (1) $x\perp_I ax+y$. Thus $x\perp_I y$ and $x\perp_I ax+y$, contradicting the uniqueness of isosceles orthogonality in strictly conver spaces proved by Kapoor and Prasad ([6], Theorem 3). Hence $(1)\Rightarrow(2)$.

We now prove $(2)\Rightarrow(1)$. If not then there exist $x,y\in X$ such that $x\perp_{L^p}y$ but $x\not\perp_I y$. We can choose $b\neq 0$ such that $x\perp_I bx+y$. But then by (2) $x\perp_{L^p}bx+y$ Thus $x\perp_{L^p}y$ and $x\perp_{L^p}bx+y$, contradicting the uniqueness of L^p-orthogonality. Hence $(2)\Rightarrow(1)$.

Corollary 6 Let $p>1$ be a fixed real number. If L^p-orthogonalty implies isosceles orthogonality in X, tnen X is a uniformly convex space.

It follows immediately from Lemma 5 and corollary 4.

Corollary 7 Let $p>1$ be a fixed real nwmber. If L^p-orthogonality is homogeneous in X, i.e., $x\perp_{L^p}y$ implies $ax\perp_{L^p}by$ for all real numbers a and b, then X is a uniformly convex space.

Theorem 8 Let $p>1$ be a fixed real number. If L^p-orthogonality is additive, *i.e.*, $x\perp_{L^p}y$ and $x\perp_{L^p}z$ implies $x\perp_{L^p}y+z$, then it is also homogeneous.

Proof Suppose L^p-orthogonality is additive and $x\perp_{L^p}y$, where x and y are arbitrary elements. Then Theorem 1 gives the existence of a number c such that $x\perp_{L^p}cx-y$. Additivity then gives $x\perp_{L^p}cx$, and hence $c=0$ if $x\neq 0$. Thus $x\perp_{L^p}-y$. Also, $y\perp_{L^p}x$ because of symmetry of L^p-orthogonality. Using additivity, we find it now follows that $nx\perp_{L^p}my$ for all integers m and n. Thus $\|nx\|^p+\|my\|^p=\|nx+my\|^p$, or $\|x\|^p+\left\|\frac{m}{n}y\right\|^p=\left\|x+\frac{m}{n}y\right\|^p$. From the continuity of $\|\cdot\|^p$ it follows that $\|x\|^p+\|ky\|^p=\|x+ky\|^p$ for all numbers k, or $x\perp_{L^p}ky$ for all k. Thus L^p-orthogonality is homogeneous if it is additive.

Corollary 9 Let $p>1$ be a fixed real number. If L^p-orthogonality is additive in X, then X is a uniformly convex space.

Remark It has been proved by Day [7] and James [5] that X is a Hilbert space further in the case of $p=2$ in these above corollaries.

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