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Vector-valued Measure and the Necessary Conditions for the Optimal Control Problems of Linear Systems*

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Abstract The vector-valued measure defined by the well-posed linear boundary value problems is discussed. The maximum principle of the optimal control problem with non-convex constraint is proved by using the vector-valued measure. Especially, the necessary conditions of the optimal control of elliptic systems is derived without the convexity of the control domain and the cost function.

Keywords optimal control, maximum principle, distributed parameter system, linear system, vector-valued measure.

Introduction

LaSalle(1960) and then many other authors (Hermes and LaSalle, 1969; Li, Xie and colleagues, 1964) have used the vector-valued measure to study time optimal control of lumped parameter systems. Li and Yao(1981a,b) and Li(1980) have used it to consider the time optimal control of distributed parameter systems. Li(1982) and then Li and Yao(1982) and Yao(1981) have extended this method to study general optimal control problems. In this paper, we further consider the role of the vector-valued measure in the study of the necessary conditions for general optimal control problems of linear systems. Especially, we prove the maximum principle of optimal control of elliptic systems without the convexity of the control domain U and the cost function f°.

Vectoe-valued measure defined by linear systems

Let Ω be an open set of space R^n with smooth boundary $\partial\Omega$. Let L be a linear differential operator in Ω and B a normal family of the boundary operators (Lions & Magenes, 1972).

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We consider the boundary value problem

$$Ly=f,\ \ x\in\Omega;\ \ By=g,\ \ x\in\partial\Omega. \tag{1}$$

Assume that the boundary value problem (1) is well-posed with $f\in\mathscr{F}$ and $g\in\mathscr{G}$, i.e., for $f\in\mathscr{F}$ and $g\in\mathscr{G}$, the bounday value problem (1) has an unique solution $y\in\mathscr{H}$, where $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ satisfy the following assumptions.

Assume that $\mathscr{F}$ satisfies the following conditions:

1) $0\in\mathscr{F}$; 2) If $f_i\in\mathscr{F}$ and $\alpha_i\in R$ $(i=1,2)$, then

$$\alpha_1 f_1+\alpha_2 f_2\in\mathscr{F};$$

3) If $f\in\mathscr{F}$, the measurable subset E of Ω is given and $f_E:\Omega\to R$ defined by

$$f_E(x)=\begin{cases} f(x) & \text{when } x\in E,\\ 0 & \text{when } x\in\Omega\setminus E,\end{cases}$$

then $f_E\in\mathscr{F}$.

For $\mathscr{G}$, we assume that $\mathscr{G}$ satisfies similar conditions as $\mathscr{F}$.

We assume that $\mathscr{H}$ is a reflexive Banach space and $|y(x)|\leqslant K\|y\|$, $\forall y\in\mathscr{H}$, for example, $\mathscr{H}$ is $H^m(\Omega)$ (Lions, 1971).

Denote the solution y of the boundary value problem (1) by

$$y=y(f,g),$$

and denote

$$y(f)=y(f,0),\ \ \mu(E)=y(f_E). \tag{2}$$

Obviously, if the subsets E_1 and E_2 of Ω are measurable, and $E_1\cap E_2=\emptyset$, then

$$\mu(E_1\cup E_2)=\mu(E_1)+\mu(E_2).$$

And, if $E_j(\subset\Omega)$ is measurable, $E_i\cap E_j=\emptyset\ (i\neq j)$, then for $E=\bigcup_j E_j$

$$\mu(E)=y(f_E)=\sum_j y(f_{E_j})=\sum_j\mu(E_j).$$

Hence, given f, $\mu(E)=y(f_E)$ defines a $\mathscr{H}$-valued measure, and it is a countable additive function on the family of all measurable subsets of Ω.

Further, we assume that the vector-valued measure $\mu(E)=y(f_E)$ has the following properties:

1) It is of bounded variation, i.e., if $E=\bigcup_j E_j$, $E_i\cap E_j=\emptyset\ (i\neq j)$, E and E_j measurable, then

$$\sum_j\|\mu(E_j)\|=\sum_j\|y(f_{E_j})\|<+\infty.$$

2) It is absolutely continuous, i.e., for $\varepsilon>0$ there exists $\delta>0$ such that

$$\|\mu(E)\|<\varepsilon,\ \ \forall\ |E|<\delta,$$

where $|E|$ denotes the Lebesgue measure of a set E in R^n.

We suppose that the $\mathscr{H}$-valued measure $\mu(E)=y(f_E)$ defined by the boundary value problem (1) is absolutely continuous.

Example The $\mathscr{H}$-valued measure defined by the Dirichlet problem

$$\Delta y = f, \ x \in \Omega; \ y = 0, \ x \in \partial\Omega \tag{3}$$

is absolutely continuous.

This can be proved by the Fredholm theory (Friedman, 1969).

Optimal control problem

1. The state equation is given by

$$Ly = f + b(x, u(x)), \ x \in \Omega; \ By = g, \ x \in \partial\Omega. \tag{4}$$

Let Z be a Banach space, U an arbitrary set of Z. Let b: $\Omega \times Z \to R$ satisfy the following assumptions:

$b(x, \cdot)$: $Z \to R$ is continuous for almost all $x \in \Omega$;

$b(\cdot, u)$: $\Omega \to R$ is Lebesgue integrable for all $u \in Z$.

If $u(\cdot)$: $\Omega \to Z$ is strongly measurable, $u(x) \in U$ for almost all $x \in \Omega$, and $b(\cdot, u(\cdot))$ Lebesgue integrable in Ω, $b(\cdot, u(\cdot)) \in \mathscr{F}$, then we call $u(\cdot)$ admissible control and denote it by

$$u(\cdot) \in U_{ad}.$$

2. Let f°: $\Omega \times R \times Z \to R$ be continuous and there exists $\partial f^\circ / \partial y$ which is continuous. For $u(\cdot) \in U_{ad}$, the unique solution of (4) is $y = y(\cdot, u)$. And we have the integral

$$J(u) = \int_\Omega f^\circ(x, y(x, u), u(x))dx. \tag{5}$$

We call $J(u)$ the cost function. The optimal control problem is to find $u^*(\cdot) \in U_{ad}$ such that

$$J(u^*) \leqslant J(u), \ \forall u(\cdot) \in U_{ad}. \tag{6}$$

We want to consider the necessary conditions of the optimal control $u^*(\cdot)$ for the problem (6).

Remark 1 We do not use the convexity of the control domain U.

Remark 2 We do not assume the convexity of the function f°.

Maximum Principle

Theorem Assume that for every given $f \in \mathscr{F}$ the $\mathscr{H}$-valued measure defined by the boundary value problem (1) is of bounded variation and absolutely continuous, $u^*(\cdot)$ is the optimal control of problem (6), $y^*(\cdot) = y(\cdot; u^*)$, and

$$-L^*\psi = \frac{\partial f^\circ(x, y^*(x), u^*(x))}{\partial y}, \ x \in \Omega; \ B^*\psi = 0, \ x \in \partial\Omega, \tag{7}$$

where L^*(and B^*) is the dual operator of L (and B). Then the maximum principle

$$\psi(x)b(x, u^*(x)) - f^\circ(x, y^*(x), u^*(x)) = \max_{u \in U}\{\psi(x)b(x, u) - f^\circ(x, y^*(x), u)\} \tag{8}$$

holds for almost all x.

For its proof we need the following lemma.

Lemma Under the assumptions of the theorem, let $u(\cdot)\in U_{ad}$ and

$$y(\cdot)=y(\cdot;u).$$

Set

$$\delta y^*(x)=y(x)-y^*(x)$$

and

$$g(x,u)=\frac{\partial f^\circ(x,y^*(x),u)}{\partial y}.$$

Then the vector-valued measure $\tilde{\mu}(E)=(\varphi_E,y_E^\circ,\tilde{y}_E)$ where

$$\varphi_E=\int_E\{g(x,u(x))-g(x,u^*(x))\}\delta y^*(x)dx,$$

$$y_E^\circ=\int_E\{f^\circ(x,y^*(x),u(x))-f^\circ(x,y^*(x),u^*(x))\}dx$$

and

$$L\tilde{y}_E=[b(\cdot,u(\cdot))-b(\cdot,u^*(\cdot))]_E,\ \ x\in\Omega;\ \ B\tilde{y}_E=0,\ \ x\in\partial\Omega$$

is of bounded variation and absolutely continuous. Proof is obvious.

Proof of Theorem

According to the lemma, $\tilde{\mu}(E)$ is absolutely continuous. And by the theorem of Uhl (1969), we have that for $\alpha\in(0,1)$ there exists a measurable subset E_α of Ω such that

$$\begin{aligned}&\Big|\alpha\int_\Omega\{g(x,u(x))-g(x,u^*(x))\}\delta y^*(x)dx\\&-\int_{E_\alpha}\{g(x,u(x))-g(x,u^*(x))\}\delta y^*(x)dx\Big|<\alpha^2,\end{aligned}\tag{9}$$

$$\begin{aligned}&\Big|\alpha\int_\Omega\{f^\circ(x,y^*(x),u(x))-f^\circ(x,y^*(x),u^*(x))\}\,dx\\&-\int_{E_\alpha}\{f^\circ(x,y^*(x),u(x))-f^\circ(x,y^*(x),u^*(x))\}dx\Big|<\alpha^2,\end{aligned}\tag{10}$$

$$\|\alpha\tilde{y}_\Omega(\cdot)-\tilde{y}_{E_\alpha}(\cdot)\|<\alpha^2.$$

Set

$$u_\alpha(x)=\begin{cases}u(x), & x\in E_\alpha,\\ u^*(x), & x\in\Omega\backslash E_\alpha.\end{cases}$$

Then $u_\alpha(\cdot)\in U_{ad}$ and

$$y(\cdot,u_\alpha)=y(\cdot,u^*)+\tilde{y}_{E_\alpha}(\cdot),\ \ \tilde{y}_\Omega(\cdot)=y(\cdot,u)-y(\cdot,u^*).$$

Therefore

$$\alpha\tilde{y}_\Omega(\cdot)-\tilde{y}_{E_\alpha}(\cdot)=\alpha y(\cdot)+(1-\alpha)y^*(\cdot)-y(\cdot,u_\alpha)$$

and hence

$$|\alpha y(x)+(1-\alpha)y^*(x)-y(x,u_\alpha)|\leqslant K\|\alpha y(\cdot)+(1-\alpha)y^*(\cdot)-y(\cdot,u_\alpha)\|\leqslant K\alpha^2.\tag{11}$$

By (10), we have

$$\int_{E_a} f^\circ(x,y^*(x),u(x))dx = \int_{E_a} f^\circ(x,y^*(x),u^*(x))dx + a\int_\Omega \{f^\circ(x,y^*(x),u(x)) - f^\circ(x,y^*(x),\ u^*(x))\}dx + O(a^2). \quad (12)$$

By (9), we have

$$\int_\Omega g(x,u_a(x))\delta y^*(x)dx = \int_{\Omega\setminus E_a} g(x,u^*(x))\ \delta y^*(x)dx + \int_{E_a} g(x,u(x))\delta y^*(x)dx$$
$$= \int_\Omega g(x,u^*(x))\delta y^*(x)dx + a\int_\Omega \{g(x,u(x)) - g(x,u^*(x))\}\delta y^*(x)dx + O(a^2)$$
$$= \int_\Omega g(x,u^*(x))\delta y^*(x)dx + O(a).$$

By (11) and the continuity of $\frac{\delta f^\circ}{\partial y}$, we have

$$J(u_a) = \int_\Omega f^\circ(x,y(x;u_a),u_a(x))dx = \int_\Omega f^\circ(x,y^*(x) + a(y(x) - y^*(x)),u_a(x))dx$$
$$+ O(a^2) = \int_\Omega f^\circ(x,y^*(x),u_a(x))dx + a\int_\Omega g(x,u^*(x))\delta y^*(x)dx + o(a). \quad (13)$$

Hence by (12) and (13), we obtain

$$J(u_a) = \int_\Omega f^\circ(x,y^*(x),u^*(x))dx + a\int_\Omega \{f^\circ(x,y^*(x),u(x)) - f^\circ(x,y^*(x),u^*(x))\}dx + a\int_\Omega g(x,u^*(x))\delta y^*(x)dx + o(a).$$

But $J(u_a) \geqslant J(u^*)$, so we have

$$a\int_\Omega \{f^\circ(x,y^*(x),u(x)) - f^\circ(x,y^*(x),u^*(x))\}dx + a\int_\Omega g(x,u^*(x))\delta y^*(x)dx + o(a) \geqslant 0.$$

Hence

$$\int_\Omega \{f^\circ(x,y^*(x),u(x)) - f^\circ(x,y^*(x),u^*(x))\}dx + \int_\Omega g(x,u^*(x))\delta y^*(x)dx \geqslant 0.$$

By the definition of $\psi(x)$, we have

$$\int_\Omega \{f^\circ(x,y^*(x),u(x)) - f^\circ(x,y^*(x),u^*(x))\}dx - \int_\Omega \{y(x) - y^*(x)\}L^*\psi(x)dx \geqslant 0.$$

Thus by the Green formula (Friedman, 1969), we have

$$\int_\Omega \psi(x)\{Ly^*(x) - Ly(x)\}dx - \int_\Omega \{f^\circ(x,y^*(x),u^*(x)) - f^\circ(x,y^*(x),u(x))\}dx \geqslant 0.$$

Hence for $u(\cdot) \in U_{ad}$, we have

$$\int_\Omega \psi(x)\{b(x,u^*(x)) - b(x,u(x))\}dx - \int_\Omega \{f^\circ(x,y^*(x),u^*(x)) - f^\circ(x,y^*(x),u(x))\}dx \geqslant 0.$$

Thus, from the above inequality and by the usual method (Li, 1980; Li and Yao, 1981a) we can prove that the maximum principle (7) holds for almost all $x \in \Omega$.

Q.E.D.

Remark 3 For the boundary control problem

$$Ly = f,\ \ x \in \Omega;\ \ By = g + \tilde{g}(x,u(x)),\ \ x \in \partial\Omega.$$

We can use the above method to derive the maximum principle.

Remark 4 For the elliptic system

$$\Delta y = f + b(x, u(x)), \quad x \in \Omega; \quad y = g, \quad x \in \partial\Omega$$

and the cost function f°, the maximum principle of optimal control holds without the conditions on the convexity of U and f°.

Remark 5 If we use the theorem 1 of Li & Yao (1982), we can obtain the maximum principle under slight conditions.

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References

[1] Friedman, A.(1969), Partial Differential Equations, Holt, Rinehart and Winston Inc., New York.

[2] Hermes, H. & J. P. LaSalle(1969), Functional Analysis and Time Optimal Control, Academic Press, New York.

[3] LaSalle, J. P.(1960), The time optimal control problem, Contributions to the Theory of Nonlinear Oscillations, V, 1-24.

[4] Li Xunjing(1980), On boundary time optimal control for systems governed by parabolic equations, *Chinese Annals of Mathematics*, 1, 453-458.

[5] Li Xunjing (1982), On applications of vector measure to the optimal control theory for distributed parameter systems, *Chinese Annals of Mathematics*, 3, 655-662.

[6] Li Xunjing, Xie Huimin and colleagues (1964), The stability of regulator systems and time optimal control, Selected Paper on Mathematics, Institute of Mathematics, Fudan University, 95-111.

[7] Li Xunjing & Yao Yunlong(1981a), Time optimal control of distributed parameter systems, *Scientia Sinica*, 24, 455-465.

[8] Li Xunjing & Yao Yunlong(1981b), On optimal control for distributed parameter systems, Proc. IFAC/81 Congress, Kyoto.

[9] Li Xunjing & Yao Yunlong(1982), On optimal control of distributed parameter systems with time lags, submitted Scientia Sinica.

[10] Lions, J. L.(1971), Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag.

[11] Lions, J.L. & E. Magenes(1972), Non-Homogeneous Boundary Value Problems and Applications, Vol. 1., Springer-Verlag.

[12] Uhl, J.J.Jr.(1969), The range of a vector-valued measure, *Proc. Amer. Math. Soc.*, 23,158-163.

[13] Yao Yunlong(1981), Vector measure and maximum principle, *Scieutia Sinica*, July, 1982, 658-666.