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The Error Estimates of Semi-discrete Galerkin Methods for some Non-linear Parabolic Equations*

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1. Introduction

Concerning the finite element analysis for linear and nonlinear parabolic equations there are a lot of papers, however, only a few of them devoted to the parabolic problems with mixed boundary conditions. In[1],[2] and [3], the finite element methods for some non-linear parabolic problems are systematically considered, but they are mainly restriced to the cases in which the boundary conditions are of Dirichlet or Neumann type. In [4], the several Galerkin procedures for parabolic problems with mixed boundary are described but their theoritical analyses are not provided.

The author considered Galerkin methods in [5], [6], [7] for linear, semi-linear and some quasi-linear diffusion equations with mixed boundary conditions and given their error estimates by virute of solutions of some linear elliptic boundary problems. In this paper, we try to improve and extend the results in [5]-[7] for more general nonlinear equations than that in [7]. In §2, the semi-discrete Galerkin procedure is given and the solvability is discussed. In §3 and §4, the optimal H^1 and L_2 error estimates by virute of space mesh parameter h are obtained respectively.

Consider the following quasi-linear parabolic equations

$$(1,1)\qquad \frac{\partial u}{\partial t} = \nabla\cdot(k(x,u)\nabla u) + \underset{\sim}{b}(x,u)\cdot\nabla u + f(x,t;u),\quad (x,t)\in\Omega\times(0,T]$$

with mixed boundary condition

$$(1,2)\qquad u(x,t) = 0,\ (x,t)\in\partial\Omega_1\times[0,T];$$

$$k(x,u)\nabla u\cdot\underset{\sim}{\nu} + \sigma(x,u)u = g(x,t;u),\ (x,t)\in\partial\Omega_2\times[0,T]$$

and initial condition

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(1,3) $$u(x,0)=u_0(x), \qquad x\in\bar{\Omega}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ and satisfying cone condition in R^n, $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$ and meas $(\partial\Omega_1)>0$; T is a fixed positive constant; $\underset{\sim}{b}(x,u)=(b_1(x,u),b_2(x,u),\cdots,b_n(x,u))$; $\underset{\sim}{\gamma}=(\gamma_1,\gamma_2,\cdots,\gamma_n)$ is the unit exterior nomal of $\partial\Omega_2$; the functions k, b_i, σ and f, g satisfy the following assumptions which will be reffered to as condition (A_1).

Condition (A_1)

(i) There exist constants k_*, $k^*>0$ such that

(1,4)
$$0<k_*\leqslant k(x,p)\leqslant k^*,\quad |b_i(x,p)|\leqslant k^*, \forall(x,p)\in\bar{\Omega}\times R, \qquad i=1,2,\cdots,n,$$
$$0\leqslant\sigma(x,p)\leqslant k^*,\quad \forall(x,p)\in\partial\Omega_2\times R.$$

(ii) k, b_i, f, σ, g, satisfy uniformly Lipschitz condition with respect to the variable p in Ω and their Lipschitz constants are denoted by the same letter L; for each $t\in[0,T]$, $f(x,t;0)\in L_2(\Omega)$ and $g(x,t,0)\in L_2(\partial\Omega_2)$; and also, f, g are continuous in variable t; inital-value function $u_0(x)\in H^1_s(\Omega)$, where

$$H^1_s(\Omega)=\{v:\ v\in H^1(\Omega),\ v|_{\partial\Omega_1}=0\}$$

and
$$H^r(\Omega)=W_{r,2}(\Omega),$$

they are usual Hilbert-Sobolev spaces, the norm on $H^r(\Omega)$ be denoted by $\|\cdot\|_r$, the subscript will be omitted in the case $r=0$.

Analogously, let $H^r(\partial\Omega)$ denote the Sobolev trace space on $\partial\Omega$ with norm $\|\cdot\|_{r,\partial\Omega}$. Specifically in the case $r=0$, $H^0(\partial\Omega)=L_2(\partial\Omega)$ and

$$\|v\|^2_{0,\partial\Omega}=\int_{\partial\Omega}v^2dS.$$

Let X be a Banach space, $\varphi(t)$ be a map: $[0,T]\to X$, define

$$\|\varphi\|_{L_p[0,T;X]}\triangleq\|\varphi\|_{L_p(X)}=\left(\int_0^T\|\varphi\|^p_X(t)dt\right)^{1/p},$$

$$\|\varphi\|_{L_\infty[0,T;X]}\triangleq\|\varphi\|_{L_\infty(X)}=\sup_{0<t<T}\|\varphi\|_X(t).$$

For convenience, we use the abbreviation nations

$$\|\varphi\|_{L_p(H^r)}\equiv\|\varphi\|_{L_p(H^r(\Omega))},\quad \|\varphi\|_{L_\infty(L_\infty)}\equiv\|\varphi\|_{L_\infty(L_\infty(\Omega))}$$

and $u(t)\equiv u(x,t)$, $b_i(u)\equiv b_i(x,u)$, $f(u)\equiv f(x;t,u)$ etc.

The weak form of problem (A) is of the following: find a map $u(t)$: $[0,T]\to H^1_s(\Omega)$ such that

(1,5)
$$\left(\frac{\partial u}{\partial t},v\right)+a(u;u;v)=(\underset{\sim}{b}(u)\cdot\nabla u,v)+(f(u),v)+\langle g(u),v\rangle \qquad \forall v\in H^1_s(\Omega),0<t\leqslant T \qquad (B)$$
$$u(0)=u_0,$$

herein
$$(w,v)=\int_\Omega wv\,d\Omega$$

(1,6) $$a(Q;w,v)=\int_{\Omega}k(x,Q)\nabla w\cdot\nabla v d\Omega+\int_{\partial\Omega_1}\sigma(Q)wvds,$$

$$\langle w,\ v\rangle=\int_{\partial\Omega_1}wvds.$$

From condition (1,4)

(1,7) $$k_*|v|_1^2\leqslant a(Q;v,v)\leqslant k^*(\|v\|_1^2+\|v\|_{0,\partial\Omega}^2),\quad\forall Q,v\in H_s^1(\Omega)$$

where the semi-norm

$$|v|_1^2=(\nabla v,\nabla v)=\sum_{i=1}^n\|v_{x_i}\|^2.$$

We shall always suppose that the solution $u(t)$ of problem (B), that is, the weak solution of problem (A), exists uniquely. Throughout this paper, we shall use letters C, C_i, ε, ε_i to denote generic constants which with different values in different inequalitys.

2 The semi-discrete Galerkin approximation and its solvability

Let $S_h(\Omega)=\text{span}\{\phi_1,\phi_2,\cdots,\phi_{N_h}\}\subset H_s^1(\Omega)$ denote a finite element subspace, where basis functions $\{\phi_i\}$ satisfy hypetheses.

(2,1) $$\phi_i\in C(\bar\Omega)\cap H_s^1(\Omega),\quad\forall h\in(0,1),\quad\|\nabla\phi_i\|_{L_\infty(\Omega)}<+\infty,\quad i=1,2,\cdots,N_h.$$

The semi-discrete Galerkin approximation for problem (A) is defined as a map $U(t)$: $[0,T]\rightarrow S_h(\Omega)$ and satisfies

(2.2) $$\left.\begin{aligned}&\left(\frac{\partial U}{\partial t},V\right)+a(U;U,V)=(\underset{\sim}{b}(U)\cdot\nabla U,V)+f(U,V)+\langle g(U),V\rangle\\&\qquad\qquad\qquad\forall V\in S_h(\Omega),\quad 0<t\leqslant T\\&U(0)=\psi_0\in S_h(\Omega),\quad\psi_0\text{ is given using appropriate procedure}^{*_1}.\end{aligned}\right\}$$

Set $U(t)=\sum_{i=1}^{N_h}\alpha_i(t)\phi_i(x)$, $\psi_0=\sum_{i=1}^{N_h}\alpha_i^{(0)}\phi_i(x)$ and substituting it into (2,1) get

(2.3) $$\left.\begin{aligned}&B\frac{d\underset{\sim}{\alpha}}{dt}+A(\underset{\sim}{\alpha})\underset{\sim}{\alpha}=E(\underset{\sim}{\alpha})\underset{\sim}{\alpha}+\underset{\sim}{F}(\underset{\sim}{\alpha})\\&\underset{\sim}{\alpha}(0)=\underset{\sim}{\alpha}^{(0)}\end{aligned}\right\}$$

where

$$\underset{\sim}{\alpha}(t)=\{\alpha_1(t)),\alpha_2(t),\cdots,\alpha_{N_h}(t)\},\quad\underset{\sim}{\alpha}^{(0)}=\{\alpha_1^{(0)},\alpha_2^{(0)},\cdots,\alpha_{N_h}^{(0)}\},$$

$$B=[(\phi_i,\phi_j)]_{N_h\times N_h},\quad A(\underset{\sim}{\alpha})=\left[a\left(\sum_1^{N_h}\alpha_s\phi_s;\ \phi_i,\ \phi_j\right)\right]_{N_h\times N_h}.$$

$$E(\underset{\sim}{\alpha})=\left[\left(\underset{\sim}{b}\sum_1^{N_h}\alpha_s\phi_s\right)\cdot\nabla\phi_j,\phi_i\right]_{N_h\times N_h}.$$

$$\underset{\sim}{F}(\underset{\sim}{\alpha})=\left\{\left(f\left(\sum_1^{N_h}\alpha_s\phi_s\right),\ \phi_i\right)\right\}_{N_h\times 1}.$$

Lemma 1 On space $H_s^1(\Omega)$, the norm $\|v\|_1$ and semi-norm $|v|_1$ are equivalent ([6], [7]);

*1 The detailed description on ψ_0 see (3,24)-(3,25) in §3.

Lemma 2 For an arbitrary fixed element $Q\in H_S^1(\Omega)$, the bilinear form $a(Q;v,w)$ is symmetrical positive definite and bounded on $H_S^1(\Omega)\times H_S^1(\Omega)$ under condition (A). ([7])

Theorem 1 Suppose that Conditions (A_1) and (2.1) hold, then the solution $U(t)$ of problem (C) is existence uniquely.

Proof Obviously, equation $(2,3)$ can be written as

$$(2,3^1)\qquad \frac{d\underset{\sim}{a}}{dt}=\underset{\sim}{G}(\underset{\sim}{a},t),\quad \underset{\sim}{a}(0)=\underset{\sim}{a}^{(0)}.$$

From assumption (A_1), every component $G_i(\underset{\sim}{a};t)$ of $\underset{\sim}{G}$ is continuous in variables $a_1,a_2,\cdots,a_{N_h}$ and t; then follows from the theory of ordinary differential equations that $a_j(t)$ $(j=1,2,\cdots,N_h)$ exist and $a_j(t)\in C^1[0,T]$.

In order to prove the uniqueness we assume that the problem (C) possesses solutions U and W. Let $\beta=U-W$ then from $(2,2)$

$$(2,4)\quad \left(\frac{\partial\beta}{\partial t},V\right)+a(U;\beta,V)=a(W;W,V)-a(U;W,V)+(\underset{\sim}{b}(U)\cdot\nabla U-b(W)\nabla W;V)$$
$$+(f(U)-f(W),V)+\langle g(U)-g(W),V\rangle,\quad \forall V\in S_h(\Omega),0<t\leqslant T.$$

From lemma 1, $(1,7)$ and $(2,4)$ with $V=\beta$, it is easy to show that there exists $k_0>0$ such that

$(*_1)$ the left-hand side of $(2,4)\geqslant\frac{1}{2}\cdot\frac{d}{dt}\|\beta\|^2+k_0\|\beta\|^2$.

Using Condition $(2,1)$ and note that $a_i(t)\in C^1[0,t]$ we see that

$$\|\nabla W\|_{L^\infty(L^\infty)}\triangleq\sup_{0<t<T}\left\{\left\|\frac{\partial W}{\partial x_1}\right\|_{L^\infty(\Omega)}(t),\cdots,\left\|\frac{\partial W}{\partial x_n}\right\|_{L^\infty(\Omega)}(t)\right\}\leqslant M_1=\text{const}.$$
$$\|W\|_{L^\infty(L^\infty(\partial\Omega_1))}\leqslant M_2=\text{const}.$$

Thus applying inequality $ab\leqslant\varepsilon a^2+\frac{1}{4\varepsilon}b^2$ $(\forall\varepsilon>0)$ and the trace inequality in $H^1(\Omega)$ we can get

$$(*_2)\quad \begin{aligned}a(W;W,\beta)-a(U;W,\beta)&\leqslant M_1L\int_\Omega\sum_i|\beta|\left|\frac{\partial\beta}{\partial x_i}\right|d\Omega+M_2L\int_\Omega\beta^2ds\\&\leqslant C\left(\varepsilon\|\beta\|_1^2+\frac{1}{4\varepsilon}\|\beta\|^3\right).\end{aligned}$$

Clearly,

$$(*_3)\qquad (f(U)-f(W),\beta)\leqslant L\|\beta\|^2;\qquad \langle g(U)-g(W),\beta\rangle\leqslant C\|\beta\|_{\frac{1}{2}}^2.$$

Applying the interpolation theory for Sobolev spaces ([10])

$$(*_4)\qquad \langle g(U)-g(W),\beta\rangle\leqslant C_2\left(\varepsilon\|\beta\|_1^2+\frac{1}{4\varepsilon}\|\beta\|^2\right)$$

$$(\underset{\sim}{b}(U)\cdot\nabla U-\underset{\sim}{b}(W)\cdot\nabla W,\beta)\leqslant k^*\int_\Omega\sum_i|\beta|\cdot|\beta_{x_i}|d\Omega+M_1L\int_\Omega\beta^2d\Omega$$

$(*_5)$ $$\leqslant C_3\left(\varepsilon\|\beta\|_1^2+\frac{1}{4\varepsilon}\|\beta\|^2\right).$$

Combining (2,4) with $(*_1)-(*_5)$, we have

$$\frac{d}{dt}\|\beta\|^2+k_0\|\beta\|_1^2\leqslant C\left(\varepsilon\|\beta\|_1^2+\frac{1}{4\varepsilon}\|\beta\|^2\right),\quad 0<t\leqslant T.$$

Choosing ε such that $C\varepsilon\leqslant k_0$ and notice that $\beta(0)=U(0)-W(0)=0$, then

$$\frac{d}{dt}\|\beta\|^2(t)\leqslant C_1\|\beta\|^2(t),\quad \beta(0)=0,$$

thus $\beta(t)\equiv 0$, the uniqueness is proved.

3 The H¹—error estimate

Let $u(t)$, $U(t)$ be the solutions of problems (B) and (C) respectively. Assume that

(3,1) $$\|u\|_{L^\infty(L^\infty(\partial\Omega))}<+\infty,\quad \|\nabla u\|_{L^\infty(L^\infty(\Omega))}<+\infty.$$

Let $W(t)$ be an arbitrary differentiable map $[0,T]\to S_h(\Omega)$ and let

$$e=u-U;\quad \eta=u-W;\quad \xi=U-W.$$

From (1,5)

$$\left(\frac{\partial W}{\partial t},V\right)+a(U;W;V)=(f(u),V)+\langle g(u),V\rangle+a(U;W,V)-a(u;u,V)$$
$$+(\underset{\sim}{b}(u)\cdot\nabla u,V)-\left(\frac{\partial\eta}{\partial t},V\right),\qquad \forall V\in S_h(\Omega),\ 0<t\leqslant T,$$

substracting this equation from (2,2) we get

$$\left(\frac{\partial\xi}{\partial t},V\right)+a(U;\xi;V)=(f(U)-f(u),V)+\langle g(U)-g(u),V\rangle+a(u;u,V)$$

(3,2)

$$-a(U;W,V)+(\underset{\sim}{b}(U)\cdot\nabla U-\underset{\sim}{b}(u)\cdot\nabla u,V)+\left(\frac{\partial\eta}{\partial t},V\right),\quad \forall V\in S_h(\Omega),0<t\leqslant T.$$

Replacing V by ξ in (3,2) then

(3,3) the left-hand side of (3,2) $\geqslant\frac{1}{2}\ \frac{d}{dt}\|\xi\|^2+k_0\|\xi\|^2$

and

(3,4) $$(f(U)-f(u),\xi)\leqslant C_1(\|\xi\|^2+\|\eta\|^2),$$

(3,5) $$\langle g(U)-g(u),\ \xi\rangle\leqslant L\int_{\partial\Omega_1}|e|\cdot|\xi|\,dS\leqslant C_2^*(\|\xi\|_{\frac{1}{2}}^2+\|\eta\|_{\frac{1}{2}}^2)$$
$$\leqslant C_2[\varepsilon\|\xi\|_1^2+\frac{1}{4\varepsilon}(\|\xi\|^2+\|\eta_1^2\|).$$

From assumption (3,1)

(3,6) $$a(u;u,\xi)-a(U;W,\xi)=a(u;u,\xi)-a(U;u,\xi)+a(U;u,\xi)-a(U;W,\xi)$$
$$\leqslant C_3\left[\varepsilon\|\xi\|_1^2+\frac{1}{4\varepsilon}(\|\xi\|^2+\|\eta\|_1^2)\right],$$

(3,7) $$(\underset{\sim}{b}(U)\cdot\triangle U-\underset{\sim}{b}(u)\cdot\nabla u,\xi)=(\underset{\sim}{b}(U)\cdot\nabla(U-W),\xi)+(\underset{\sim}{b}(U)\cdot\nabla(W-u),\xi)$$
$$+((b(u)-b(U))\cdot\nabla u,\xi)\equiv I_1+I_2+I_3.$$
$$I_1\leqslant C_4^*\Big((\varepsilon\|\xi\|_1^2+\frac{1}{4\varepsilon}\|\xi\|^2\Big),$$
$$I_2\leqslant C_5^*(\|\eta\|_1^2+\|\xi\|^2).$$

From (3,1)
$$I_3\leqslant C_6^*(\|\xi\|^2+\|\eta\|^2).$$

Therefore

(3,8) $$(\underset{\sim}{b}(U)\cdot\nabla U-\underset{\sim}{b}(u)\cdot\nabla u,\xi)\leqslant C_4\Big\{\varepsilon\|\xi\|_1^2+\frac{1}{4\varepsilon}(\|\xi\|^2+\|\eta\|_1^2)\Big\}.$$

And also,

(3,9) $$\Big(\frac{\partial\eta}{\partial t},\xi\Big)\leqslant\Big\|\frac{\partial\eta}{\partial t}\Big\|_{-1}\cdot\|\xi\|_1\leqslant\varepsilon\|\xi\|_1^2+\frac{1}{4\varepsilon}\Big\|\frac{\partial\eta}{\partial t}\Big\|_{-1}^2.$$

Combining (3,2) with (3,3), (3,9) we obtain

(3,10) $$\frac{1}{2}\frac{d}{dt}\|\xi\|^2+k_0\|\xi\|_1^2\leqslant C\Big\{\varepsilon\|\xi\|_1^2+\frac{1}{4\varepsilon}\Big(\|\xi\|^2+\|\eta\|_1^2+\Big\|\frac{\partial\eta}{\partial t}\Big\|_{-1}^2\Big)\Big\},$$

where C dependent on $\|u\|_{L_\infty(L_\infty(\partial\Omega))}$, $\|\nabla u\|_{L_\infty(L_\infty(\Omega))}$ but it is independent of h and W. Choosing ε sufficiently small then

(3,11) $$\frac{d}{dt}\|\xi\|^2+\|\xi\|_1^2\leqslant C_1\Big\{\|\xi\|^2+\|\eta\|_1^2+\Big\|\frac{\partial\eta}{\partial t}\Big\|_{-1}^2\Big\}.$$

Integrating (3,11) from $t=0$ to $t=\tau$ and written variable τ as t, applying Gronwall inequality, then

(3,12) $$\|\xi\|^2(t)+\int_0^t\|\xi\|_1^2(\tau)d\tau\leqslant C_2\Big\{\|\xi\|^2(0)+\|\eta\|_{L_2(H^1)}^2+\Big\|\frac{\partial\eta}{\partial t}\Big\|_{L_2(H^{-1})}^2\Big\}.$$

Therefore
$$\|\xi\|_{L_\infty(L_2)}+\|\xi\|_{L_2(H^1)}^2\leqslant C_2\{\|\xi\|^2(0)+\|\eta\|_{L_2(H^1)}^2+\Big\|\frac{\partial\eta}{\partial t}\Big\|_{L_2(H^{-1})}^2\Big\}.$$

By the triangle inequality

(3,13) $$\|e\|_{L_\infty(L_2)}+\|e\|_{L_2(H^1)}\leqslant C\Big\{\|\xi\|(0)+\|\eta\|_{L_\infty(L_2)}+\|\eta\|_{L_2(H^1)}+\Big\|\frac{\partial\eta}{\partial t}\Big\|_{L_2(H^{-1})}\Big\}.$$

To sum up, we have

Theorem 2 Suppose that condition (A_1) and (3,1) hold, then there exists a constant C which is independent of h such that for every differentiable map $W(t)$: $[0,T]\to S_h(\Omega)$, the following estimate is true:

(3,14) $$\|u-U\|_{L_\infty(L_2)}+\|u-U\|_{L_2(H^1)}\leqslant C\Big\{\|U(0)-W(0)\|+\|u-W\|_{L_\infty(L_2)}$$
$$+\|u-W\|_{L_2(H^1)}+\Big\|\frac{\partial(u-W)}{\partial t}\Big\|_{L_2(H^{-1})}\Big\}.$$

To estimate the approximation order for error u-U, we make further assumptions which will be reffered to as

Condition (A_2)

(i) Conditions (2,1) and (3,1) holds;

(ii) $u\in L_2(H^r)$, $\frac{\partial u}{\partial t}\in L_2(H^{r-1})$, $u_0\in H^{r-1}(\Omega)$, where $r\geqslant 2$;

(iii) $S_h(\Omega)$ is taken from a family of finite element space of class $\tilde{S}_{1,r}(\Omega), r\geqslant 2$, that is, $S_h(\Omega)\subset H^1_S(\Omega)$ and there exists a constant $C>0$ such that for each $v\in H^1_S(\Omega)\cap H^l(\Omega)$, $p\leqslant l\leqslant r$, $p=0,1$, the following inequality holds

(3,15) $$\inf_{\chi\in S_h(\Omega)}\|v-\chi\|_p\leqslant Ch^{l-p}\|v\|_l.$$

(iv) Boundary $\partial\Omega$ is regular enough such that for every $\psi\in H^1(\Omega)$ the boundary-value problem

(3,16) $$\left.\begin{aligned} &-\Delta\varphi+\varphi=\psi \qquad \chi\in\Omega \\ &\varphi\Big|_{\partial\Omega_1}=0; \qquad \frac{\partial\varphi}{\partial\gamma}\Big|_{\partial\Omega_2}=0 \end{aligned}\right\}$$

possesses an unique weak solution $\varphi\in H^3(\Omega)\cap H^1_S(\Omega)$*) which obeys the priori-estimate

(3,17) $$\|\varphi\|_3\leqslant C\|\psi\|_1.$$

Now, we take $W(t)$ to be H^1-projection into $S_h(\Omega)$ of $u(t)$ at each $t\in[0,T]$, that is, $W(t)$: $[0,T]\to S_h(\Omega)$ and satisfies

(3,18) $$(\nabla(W-u),\nabla V)+(W-u,V)=0,\quad \forall V\in S_h(\Omega).$$

Obviously, $W(t)$ defined by (3,18) exists uniquely, moreover, $W(t)$ is differentiable in variable t as well as $u(t)$.

From Conditions (A_2) (ii), (iii) and definition of $W(t)$

$$\|\eta\|_1(t)=\|u-W\|_1(t)=\inf_{\chi\in S_h(\Omega)}\|u(t)-\chi\|_1\leqslant Ch^{r-1}\|u\|_r(t).$$

Thus

(3,19) $$\|\eta\|_{L_2(H^1)}\leqslant Ch^{r-1}\|u\|_{L_2(H^r)}.$$

Since $\frac{\partial W}{\partial t}$ is H^1-projection into $S_h(\Omega)$ of $\frac{\partial u}{\partial t}$, from Condition (A_2) (ii), (iii),

(3,20) $$\left\|\frac{\partial\eta}{\partial t}\right\|_1\leqslant Ch^{r-2}\left\|\frac{\partial u}{\partial t}\right\|_{r-1} \qquad (\forall t\in(0,T])$$

It is easy to prove ([9])

Lemma 3 Let $W(t)$: $[0,T]\to S_h(\Omega)$ be a map defined by (3,18), then under condition (A_2)

(3,21) $$\left\|\frac{\partial\eta}{\partial t}\right\|_{L_2(H^{-1})}\leqslant Ch^r\left\|\frac{\partial u}{\partial t}\right\|_{L_2(H^{r-1})}.$$

Notice that

$$\|\eta\|^2(t)=\|\eta\|^2(0)+2\int_0^t\left(\frac{\partial\eta}{\partial t},\eta\right)d\tau$$

*) Obviously, φ exists and is unique. If Ω is convex-polygon or C^3-domain, inequality (3,17) is obeyed ([6]).

and

$$\left(\frac{\partial\eta}{\partial t},\eta\right)\leqslant\left\|\frac{\partial\eta}{\partial t}\right\|_{-1}\cdot\|\eta\|_1.$$

Using inequalites (3,19) and (3,20) we have

(3,22) $$\|\eta\|_{L_\infty(L_2)}\leqslant Ch^{r-1}.$$

Choosing $U(0)=\psi_0\in S_h(\Omega)$ such that

(3,23) $$\|\xi\|(0)=\|U(0)-W(0)\|=\|\psi_0-W(0)\|\leqslant Ch^{r-1}.$$

To the end, it is sufficient to take $\psi_0=W(0)$, here $W(0)$ is defined by

$$(\nabla(W(0)-u_0),\nabla V)+(W(0)-u_0,V)=0,\quad\forall V\in S_h(\Omega).$$

For simplicity, we can take ψ_0 to be L_2-projection into $S_h(\Omega)$ of u_0, that is,

$$(\psi_0-u_0,V)=0,\qquad\forall V\in S_h(\Omega).$$

From inequalitys (3,13), (3,19), (3,21)—(3,22), we obtain immediately the following

Theorem 3 Suppose that Conditions (A_1) and (A_2) hold. If initial value function $U(0)\equiv\psi_0$ is choosen such that inequality (3,22) hold then the optimal H^1-estimate

(3,25) $$\|u-U\|_{L_\infty(L_2)}+\|u-U\|_{L_2(H^1)}\leqslant Ch^{r-1}$$

is true, where C is independent of h and U.

4 The L_2-estimate for error u-U

We now wish to develop an L_2-estimate for error u-U.

Condition (A_3)

(i) Conditions (i) (ii) and (iii) in (A_2) hold; moreover, $u\in L_\infty(H^r)$

(ii) $k(\cdot,u(\cdot,t)))_t$, $(k(\cdot,u(\cdot,t)))_{tt}$ and $(k(x,u(x,t)))_{x_it}\in L_\infty(\Omega\times[0,T])$, $(\sigma(\cdot,u(\cdot,t)))_t\in L_\infty(\partial\Omega_2\times[0,T])$; $\frac{\partial}{\partial x_i}b_i(x,u(x,t))\in L_\infty(\Omega\times[0,T])$, $i=1,2,\cdots n$.

(iii) For each $\psi\in H^l(\Omega)$ and each $\gamma\in H^{l+\frac{1}{2}}(\partial\Omega)$, $l=0,1$, the linear problem

(4,1) $$a(u(t);\phi,v)=(\psi,v)+\langle\gamma,v\rangle,\quad\forall v\in H_s^1(\Omega),(\forall t\in[0,T])$$

exists unique solution $\phi\in H^{l+2}(\Omega)\cap H_s^1(\Omega)$*) which satisfies the reqularity estimate

(4,2) $$\|\phi\|_{l+2}\leqslant C\{\|\psi\|_l+\|\gamma\|_{l+\frac{1}{2},\partial\Omega}\},$$

here C is independent of ψ and ϕ.

Now we take $W(t)$ to be Galerkin projection into $S_h(\Omega)$ of $u(t)$, that is, $W(t)$: $[0,T]\to S_h(\Omega)$ and satisfies

(4,3) $$a(u(t);W(t),V)=a(u(t);u(t),V),\quad\forall V\in S_h(\Omega).$$

Using Lemma 2 and Lax-Milgram theorem, we see that $W(t)$ exists uniquely

*) From Lemma 2 and Lax-Milgram theorem, we see immediately that the solution of problem (4,1) exists uniquely.

and it is a differentiable map $[0,T]\to S_h(\Omega)$.

Let $\eta=u-W$, $\xi=U-W$ again. First at all, we estimate norms

$$\|\eta\|_{L_\infty(L_2)},\quad \|\eta\|_{L_\infty}(H^{-\frac{1}{2}}(\partial\Omega)),\quad \left\|\frac{\partial\eta}{\partial t}\right\|_{L_2(H^{-1})}.$$

Since $a(u(t);\cdot,\cdot)$ is positive definite and bounded,

$$\|\eta\|_1(t)\leqslant\beta\inf_{\chi\in S_h(\Omega)}\|u-\chi\|_1(t)\leqslant Ch^{r-1}\|u\|_r(t).$$

It is follows from Condition (A_3) (iii) and applying Nitsche method that

$$\|\eta\|(t)\leqslant C_1h\|\eta(t)\|_1\leqslant Ch^r\|u\|_r(t)$$

and

$$\|\eta\|_{L_\infty(L_2)}\leqslant Ch^r\|u\|_{L_\infty(H^r)}. \tag{4,4}$$

Using the technique which is used to prove Lemma 4 in [8] we can prove

Lemma 4 Let $S_h(\Omega)\subset\tilde{S}_{1,r}(\Omega)$, $u\in L_p(H^r)$, $p=2,+\infty$, and $W(t)$ be the solution of problem (4,3) then under Conditions (A_1) (i) and (A_3) (iii) we have

$$\|\eta\|_{L_p(H^{-\frac{1}{2}}(\partial\Omega))}=\|u-W\|_{L_p(H^{-\frac{1}{2}}(\partial\Omega))}\leqslant Ch^r\|u\|_{L_p(H^r)},\quad p=2,+\infty. \tag{4,5}$$

The detailed proof for Lemma 4 is omitted (see [8]).

In order to estimate the norm $\left\|\frac{\partial\eta}{\partial t}\right\|_{L_2(H^{-1})}$, differentiating (4,3) with respect to t yields

$$a\left(u(t);\frac{\partial\eta}{\partial t},V\right)=-\int_\Omega\frac{\partial k}{\partial t}\nabla\eta\cdot\nabla Vd\Omega-\int_{\partial\Omega}\frac{\partial\sigma}{\partial t}\eta\cdot VdS. \tag{4,6}$$

Taking V to be $\chi-\frac{\partial W}{\partial t}$, here χ is an arbitiary element in $S_h(\Omega)$, then

$$a\left(u(t);\frac{\partial\eta}{\partial t},\frac{\partial\eta}{\partial t}\right)=-a\left(u(t);\frac{\partial\eta}{\partial t},\chi-\frac{\partial W}{\partial t}\right) \tag{4,7}$$
$$-\int_\Omega\frac{\partial k}{\partial t}\nabla\eta\cdot\nabla\left(\chi-\frac{\partial W}{\partial t}\right)d\Omega-\int_{\partial\Omega}\frac{\partial\sigma}{\partial t}\cdot\eta\left(\chi-\frac{\partial W}{\partial t}\right)dS.$$

Using Condition (A_3) (ii) and the trace inequality, we get

$$k_0\left\|\frac{\partial\eta}{\partial t}\right\|_1^2\leqslant C\left\{\varepsilon\left\|\frac{\partial\eta}{\partial t}\right\|_1^2+\frac{1}{4\varepsilon}\left(\left\|\chi-\frac{\partial u}{\partial t}\right\|_1^2+\|\eta\|_1^2\right)\right\},$$

choosing ε sufficiently small then

$$\left\|\frac{\partial\eta}{\partial t}\right\|_1\leqslant C\left\{\|\eta\|_1+\inf_{\chi\in S_h(\Omega)}\left\|\frac{\partial u}{\partial t}-\chi\right\|_1\right\}\leqslant C_1h^{r-2}\left(\|u\|_r+\left\|\frac{\partial u}{\partial t}\right\|_{r-1}\right) \tag{4,8}$$
$$\forall t\in[0,T].$$

Let ψ be an arbitrary element of $H^1(\Omega)$, $\varphi(t)$ be the solution to the following problem

$$a(u(t);\varphi(t),v)=(\psi,v),\quad\forall v\in H^1_2(\Omega).$$

Choosing $v=\partial\eta/\partial t$, then

$$\left(\frac{\partial\eta}{\partial t},\psi\right)=a\left(u(t);\varphi-\chi,\frac{\partial\eta}{\partial t}\right)+a\left(u(t);\chi,\frac{\partial\eta}{\partial t}\right)=I_1+I_2,\quad\chi\in S_h(\Omega), \tag{4,9}$$

where

$$I_1 = a\left(u(t);\varphi-\chi,\frac{\partial\eta}{\partial t}\right) \leqslant C_1^*\left\|\frac{\partial\eta}{\partial t}\right\|_1 \|\varphi-\chi\|_1.$$

Noting that $\varphi\in H^3(\Omega)\cap H_S^1(\Omega)$ and from (4,8)

$$I_1 \leqslant C_1 h^{r-2}h^2\|\varphi\|_3 \leqslant C_2 h^r\|\psi\|_1. \tag{4,10}$$

In addition,

$$I_2 \leqslant a\left(u(t);\chi,\frac{\partial\eta}{\partial t}\right) = a\left(u(t),\frac{\partial\eta}{\partial t},\chi\right) = -\int_\Omega \frac{\partial k}{\partial t}\nabla\eta\cdot\nabla\chi d\Omega$$
$$-\int_{\partial\Omega_1}\frac{\partial\sigma}{\partial t}\eta\chi dS = -\int_\Omega \frac{\partial k}{\partial t}\nabla\eta\cdot\nabla(\chi-\varphi)d\Omega - \int_{\partial\Omega}\frac{\partial\sigma}{\partial t}\eta(\chi-\varphi)dS$$
$$-\int_\Omega\frac{\partial k}{\partial t}\nabla\eta\cdot\nabla\varphi d\Omega - \int_{\partial\Omega_1}\frac{\partial\sigma}{\partial t}\eta\varphi dS \equiv J_1+J_2+J_3+J_4.$$

Obviously,

$$J_1 \leqslant C_1^*\|\eta\|_1\cdot\|\chi-\varphi\|_1. \tag{Δ_1}$$

After integration by parts we have

$$J_3 = -\int_\Omega \frac{\partial k}{\partial t}\nabla\eta\cdot\nabla\varphi d\Omega \leqslant C_2^*\|\eta\|\cdot\|\varphi\|_2 + C_3^*\|\eta\|\cdot\|\varphi\|_1 + \int_{\partial\Omega_1}\sum_{i=1}^n \frac{\partial k}{\partial t}\eta\frac{\partial\varphi}{\partial x_i}\gamma_i dS.$$

Applying the duality of $H^{\frac12}(\partial\Omega)$ and $H^{-\frac12}(\Omega)$

$$\int_{\partial\Omega_1}\sum_{i=1}^n\frac{\partial k}{\partial t}\eta\frac{\partial\varphi}{\partial x_i}\gamma_i dS \leqslant C_4^*\|\eta\|_{-\frac12,\partial\Omega}\|\nabla\varphi\|_{\frac12,\partial\Omega} \leqslant C_5^*\|\eta\|_{-\frac12,\partial\Omega}\|\nabla\varphi\|_1 \leqslant C_6^*\|\eta\|_{-\frac12,\partial\Omega}\|\varphi\|_2.$$

Hence

$$J_3 \leqslant C_7^*(\|\eta\| + \|\eta\|_{-\frac12,\partial\Omega})\|\varphi\|_2. \tag{Δ_2}$$

Like wise,

$$J_2 \leqslant C_8^*(\|\eta\|_{-\frac12,\partial\Omega}\cdot\|\varphi-\chi\|_1), \tag{Δ_3}$$

$$J_4 \leqslant C_9^*\|\eta\|_{-\frac12,\partial\Omega}\|\varphi\|_1 \leqslant C_{10}^*\|\eta\|_{-\frac12,\partial\Omega}\|\varphi\|_3.$$

Combining (4,6) with (Δ_1), (Δ_2), (Δ_3), using Lemma 4 and note that

$$\inf_{\chi\in S_h(\Omega)}\|\varphi-\chi\|_1 \leqslant Ch^2\|\varphi\|_3 \leqslant C_1h^2\|\psi\|,$$

we have

$$I_2 \leqslant C_3h^r\|\varphi\|_1. \tag{4,11}$$

Substituting (4,10), (4,11) into (4,9) yields

$$\left(\frac{\partial\eta}{\partial t},\psi\right) \leqslant Ch^r\|\psi\|_1 \qquad \forall\psi\in H^1(\Omega).$$

Therefore we get

Lemma 5 Let $u(t)$ and $W(t)$ be the solutions for problems (B) and (4,3) respectively, then under Conditions (A_1) and (A_3)

$$\left\|\frac{\partial(u-W)}{\partial t}\right\|_{L_2(H^{-1})} = \left\|\frac{\partial\eta}{\partial t}\right\|_{L_2(H^{-1})} \leqslant Ch^r, \tag{4,12}$$

where constant C is independent of h and W.

To derive the L_2-estimate of u-U, further assumption is required yet.

Condition (A_4) There exists constant K which is independent of h such that for each $h\in(0,1)$, the solution $W(t)$ for problem (4,3) satisfies

(4,13) $$\|\nabla W\|_{L_\infty(L_\infty)}\leqslant K.$$

It is able to show that under Condition (A_3) (i), for considerable general finite element space satisfying inverse property, condition (4,13) holds (see[9]), specific examples can be found in [3].

Using the same treatment as in §3, we have

(4,14) $$\left(\frac{\partial\xi}{\partial t},V\right)+a(U;\xi,V)=(f(U)-f(u),V)+\langle g(U)-g(u),V\rangle+a(u;u,V)-a(U,W,V)+(\underset{\sim}{b}(U)\cdot\nabla U-\underset{\sim}{b}(u)\nabla u,V)+\left(\frac{\partial\eta}{\partial t},V\right).$$

Choosing $V=\xi$, reserving estimates (3,3), (3,4), (3,9) in §3 and anew estimate the terms 2-nd, 3-rd and 4-th on the right hand side of (4,14).

Indeed, it is easy to see that

(4,15) $$\langle g(U)-g(u),\xi\rangle\leqslant C_1\left\{\varepsilon\|\xi\|_1^2+\frac{1}{4\varepsilon}(\|\xi\|^2+\|\eta\|^2_{-\frac{1}{2},\partial\Omega})\right\}.$$

From Condition (A_4),

$$a(u;u,\xi)-a(U;W,\xi)=a(u;W,\xi)-a(U;W,\xi)$$
$$\leqslant C_2^*\left\{\varepsilon\|\xi\|_1^2+\frac{1}{4\varepsilon}(\|\eta\|^2+\|\xi\|^2)\right\}+\left|\int_{\partial\Omega_1}[\sigma(U)-\sigma(u)]\,W\xi dS\right|.$$

Now

$$\left|\int_{\partial\Omega_1}[\sigma(U)-\sigma(u)]W\xi dS\right|=\left|-\int_{\partial\Omega_1}[\sigma(U)-\sigma(u)]\eta\xi dS+\int_{\partial\Omega_1}[\sigma(U)-\sigma(u)]u\xi dS\right|$$
$$\leqslant C_3^*\left\{\varepsilon\|\xi\|_1^2+\frac{1}{4\varepsilon}(\|\xi\|^2+\|\eta\|^2_{-\frac{1}{2},\partial\Omega})\right\}.$$

Hence

(4,16) $$a(u;u,\xi)-a(U;W,\xi)\leqslant C_2\left\{\varepsilon\|\xi\|_1^2+\frac{1}{4\varepsilon}(\|\xi\|^2+\|\eta\|^2+\|\eta\|^2_{-\frac{1}{2},\partial\Omega})\right\}.$$

Set

(4,17) $$J=(\underset{\sim}{b}(U\cdot\nabla U-\underset{\sim}{b}(u)\cdot\nabla u,\xi)=(\underset{\sim}{b}(U)\cdot(\nabla U-\nabla W),\xi)+(\underset{\sim}{b}(U)\cdot(\nabla W-\nabla u),\xi)+((\underset{\sim}{b}(U)-b(u))\cdot\nabla u,\xi)=J_1+J_2+J_3.$$

Obviously

$(*_1)$ $$J_1\leqslant\tilde{C}_1\left(\varepsilon\|\xi\|_1^2+\frac{1}{4\varepsilon}\|\xi\|^2\right).$$

Since $$\|\nabla u\|_{L_\infty(L_\infty)}<+\infty,$$

$(*_2)$ $$J_3\leqslant\tilde{C}_3(\|\xi\|^2+\|\eta\|^2).$$

And also,

$$J_2=-(\underset{\sim}{b}(U)\cdot\nabla\eta,\xi)=([\underset{\sim}{b}(u)-\underset{\sim}{b}(U)]\cdot\nabla\eta,\xi)-(\underset{\sim}{b}(u)\cdot\nabla\eta,\xi)=Q_1+Q_2.$$

Using condition (A_4) we have $\|\nabla\eta\|_{L_\infty(L_\infty)}<+\infty$, thus

$(*_3)$ $$Q_1\leqslant C_1^*(\|\xi\|^2+\|\eta\|^2).$$

Integrating by parts for term $Q=-(\underset{\sim}{b}(u)\cdot\nabla\eta,\xi)$ and applying the same treatment as used to deriving inequality (4,15), we can get an estimate for Q_2; then combining it with $(*_3)$ we have

$$(*_4)\qquad J_2\leqslant\widetilde{C}_2\left\{\varepsilon\|\xi\|_1^2+\frac{1}{4\varepsilon}(\|\xi\|^2+\|\eta\|^2+\|\eta\|^2_{-\frac{1}{2},\partial\Omega})\right\}.$$

Therefore

$$(\underset{\sim}{b}(U)\cdot\nabla U-b(u)\cdot\nabla u,\xi)\leqslant C_3\left\{\varepsilon\|\xi\|_1^2+\frac{1}{4\varepsilon}(\|\xi\|^2+\|\eta\|^2+\|\eta\|^2_{-\frac{1}{2},\partial\Omega})\right\}.$$

Add up, we obtain

$$(4,19)\qquad \frac{d}{dt}\|\xi\|^2+\|\xi\|_1^2\leqslant C\left\{\|\xi\|^2+\|\eta\|^2+\|\eta\|^2_{-\frac{1}{2},\partial\Omega}+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{-1}\right\}.$$

Applying Gronwall inequality and triangle inequality we have

$$(4,20)\qquad \|e\|^2_{L_\infty(L_2)}\leqslant C\left\{\|\eta\|^2_{L_\infty(L_2)}+\left\|\frac{\partial\eta}{\partial t}\right\|^2_{L_2(H^{-1})}+\|\eta\|^2_{L_2(H^{-\frac{1}{2}}(\partial\Omega))}+\|\xi\|^2(0)\right\}.$$

Using (4,4), Lemma 4 and Lemma 5 then

$$(4,21)\qquad \|\eta\|_{L_\infty(L_2)}+\left\|\frac{\partial\eta}{\partial t}\right\|_{L_2(H^{-1})}+\|\eta\|_{L_2(H^{-\frac{1}{2}}(\partial\Omega))}\leqslant Ch^r.$$

Choosing initial value function $U(0)=\psi_0\in S_h(\Omega)$ such that

$$(4,22)\qquad \|\xi(0)\|=\|\psi_0-W(0)\|\leqslant C_1h^r,$$

where $W(0)$ is solution of problem (4,3) at $t=0$. From (4,20),(4,21) and (4,22)

$$(4,23)\qquad \|e\|_{L_\infty(L_2)}=\|u-U\|_{L_\infty(L_2)}\leqslant Ch^r.$$

To sum up, we obtain the final result

Theorem 4 Let u, U be the solutions for probleme (B) and (C) respectively, if initial value function $U(0)=\psi_0$ satisfies inequality (4,22) then under conditions (A_1), (A_3) and (A_4), error u-U possesses optimal L_2-approximating order with respect to h, that is

$$\|u-U\|_{L_\infty(L_2)}\leqslant Ch^r$$

where C is a constant independent of h and U.

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