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On the Uniqueness Theorems for the Closed Convex Hypersurfaces in a Space of Constant Curvature*

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In this paper we generalized the uniqueness theorem of Alexadroff-Fenchel-Jessen, the Cohn-Vossen theorem and the Hilbert-Liebmann-Hsiung theorem to hypersurfaces in a sphere S^{n+1} or a hyperbolic space H^{n+1}.

First of all, we generalized the uniqueness theroem of Alexandroff-Fenchel-Jessen as follows:

Theorem 1 Let Σ, Σ' be two closed, strictly convex hypersurfaces in a Euclidean space E^{n+1}, $f:\Sigma\to\Sigma'$ be a diffeomorphism such that $\mathrm{III}=f^*\mathrm{III}'$, III and III' being the third fundamental forms of Σ and Σ' respectively. If for a fixed r, $P_r=P'_r$, then f is a rigid motion(including a possible reflection), where p_r(resp. P'_r) is the r-th elementary symmetric function of the principal radii of curvature of Σ (resp. Σ').

We derived some integral formulas for a pair of hypersurfaces in S^{n+1} or H^{n+1} by which we proved some uniqueness theorems. Our main results are:

Theorem 2 Let Σ, Σ' be two closed, strictly convex hypersurfaces in S^{n+1}, f: $\Sigma\to\Sigma'$ be a diffeomorphism such that $\mathrm{III}=f^*\mathrm{III}'$.

1) When $n\geqslant 3$ f is a motion.

2) When $n=2$, if $P_1\geqslant P'_1$ then f is a motion.

Theorem 3 Let Σ, Σ' be two closed, strictly convex hypersurfaces in S^{n+1} f: $\Sigma\to\Sigma'$ be an isometry.

1) When $n\geqslant 3$ f is a motion.

2) When $n=2$, if $S_1\geqslant S'_1$ then f is a mosion.

Theorem 4 Let Σ be a closed, strictly convex hypersurface in S^{n+1}. If $S_r=const.$ for a fixed r, $1\leqslant r\leqslant n$,then Σ is a sphere.

Theorem 5 Let Σ be a closed, strictly convex hypersurface in S^{n+1}; If

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$\frac{S_r}{S_\tau}$ = const. for two integer r and $\tau, 1 \leqslant \tau < r \leqslant n$, then Σ is a sphere.

Let H^{n+1} be the $(n+1)$-disc $x_1^2 + \cdots + x_{n+1}^2 < 1$ in Euclidean space E^{n+1}, define the hyperbolic metric in H^{n+1} as follows:

$$g(V,W) = \frac{4V \cdot W}{(1-\|x\|^2)^2}, \qquad \begin{array}{l} V,\ W \in H_x^{n+1} \\ x = (x_1, \cdots, x_{n+1}) \in H^{n+1}. \\ \|x\|^2 = x \cdot x. \end{array}$$

where the dot product on the right hand side is the usual dot product on E^{n+1}. it is well known that H^{n+1} is a space of constant curvature -1. We proved.

Theorem 6 Let Σ, Σ' be two closed, local strictly convex hypersurfaces in H^{n+1} $f: \Sigma \to \Sigma'$ be a diffeomorphism such that $\mathrm{III} = f^* \mathrm{III}'$.

1) When $n \geqslant 3$, Σ and Σ' differ by a motion in H^{n+1}.

2) When $n = 2$. If $P_1 \geqslant P_1'$ then Σ and Σ' differ by a motion in H^3.

Theorem 7 Let Σ be a closed, local strictly convex hypersurface in H^{n+1}. If P_r = const. for a fixed $r, 1 \leqslant r \leqslant n$, tnen Σ is totally umbilical.

Theorem 8 Let Σ be a closed, local strictly convex hypersurface in H^{n+1}. If

$\frac{P_r}{P_\tau}$ = const. for two integers r anc $\tau, 1 \leqslant \tau < r \leqslant n$, then Σ is totally umbilical.

Theorem 9 Let Σ, Σ' be two closed, local strictly convex hypersurfaces in H^{n+1}, $f: \Sigma \to \Sigma'$ be an isometry. Suppose that $(0,0,\cdots,0)$ is an interior point of Σ, and $y_{n+1} > 0$ everywhere, where $y_{r+1} = g(x, e_{n+1})$ is the support funcfion of Σ.

1) When $n \geqslant 3$ Σ and Σ' differ by a motion in H^{n+1}.

2) When $n = 2$. If $S_1 \geqslant S_1'$ then Σ and Σ' differ by a motion in H^3.

Theorem 10 Let Σ be a closed, local, strictly convex hypersurface in H^{n+1}. Suppose that $(0,0,\cdots,0)$ is an interior point of Σ and $y_{n+1} > 0$ everywhere. If $S_r = const.$ for a iixed r, $1 \leqslant r \leqslant n$, then Σ is totally umbilical.

Theorem 11 Let Σ be a closed, local strictly convex hypersurface in H^{n+1}, If $\frac{S_r}{S_\tau}$ = const. for two fixed integers r and τ, $1 \leqslant \tau < r \leqslant n$, then Σ is totally umbilical.

Let Σ be a closed hypersurface in S^{n+1}, there are two global unit normal vector fields $e_{n+1}(x)$ and $e_{n+2}(x)$ over Σ, where $e_{n+1}(x)$ is tangent to S^{n+1}, and $e_{n+2}(x)$ is parallel to the radius vector field of S^{n+1}. The map

$$G: \Sigma \to S^{n+1}$$
$$x \longmapsto e_{n+1}(x)$$

is called the Gauss map.

Theorem 12 Let Σ be a closed, strictly convcx hypersurface in S^{n+1}, G be the Gauss map. Put $G(\Sigma) = \Sigma^*$, we have

1) $G:\Sigma\to\Sigma^*$ is a diffeomorphism and $G^2=I_d$.

2) Σ^* is a closed, strictly convex hypersurface in S^{n+1}

3) $\mathrm{I}^*=\mathrm{III}$,

$\mathrm{II}^*=\mathrm{II}$,

$\mathrm{III}^*=\mathrm{I}$.

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