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The Algebra of Pseudo-Differential Operator on the Functional Space $W_\lambda S^m_{\rho,\delta}$

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For popularizing the functional space $S^m_{\rho,\delta}$ which is common in use, a Frechet functional space $W_\lambda S^m_{\rho,\delta}$ is defined in this paper and an exploration is attempted on the algebraic characteristics of the pseudo-differential operators stipulated by the functional space $W_\lambda S^m_{\rho,\delta}$.

Definition 1 We say $P(x,\xi)\in C^\infty(R^n_x\times R^n_\xi)$ is a symbol of the functional set $W_\lambda S^m_{\rho,\delta}$ $(-\infty<m<+\infty,\ 0\leqslant\delta\leqslant\rho\leqslant 1,\ \delta<1)$, if $P(x,\xi)$ has the property that for any multi-indices $\alpha,\beta\in N^n_0$ there exists a constant $C_{\alpha,\beta}$ such that

$$|P^{(\alpha)}_{\beta}(x,\xi)|\equiv|\partial^\alpha_\xi D^\beta_x P(x,\xi)|\leqslant C_{\alpha,\beta}[\lambda(\xi)]^{m+\delta|\beta|-\rho|\alpha|}\qquad \xi\in R^n_\xi$$

in which the function $\lambda(\xi)$ must satisfy the conditions below:

$$(*)\qquad\begin{cases}\langle\xi\rangle\equiv(1+|\xi|^2)^{\frac12}\leqslant\lambda(\xi)\leqslant K\langle\xi\rangle^{2m_0} & \xi\in R^n_\xi\\ |\lambda(\xi)-\lambda(\xi')|\leqslant N|\xi-\xi'| & \xi,\xi'\in R^n_\xi\end{cases}$$

here $k>0$, $N>0$, $\frac{1}{2}<m_0<\frac{1}{2\delta}$ are constants.

Assume $P(x,\xi)\in W_\lambda S^m_{\rho,\delta}$, its semi-norms will be defined by

$$|P|^{(m)}_l=\max_{|\alpha+\beta|\leqslant l}\sup_{x,\xi}|P^{(\alpha)}_{\beta}(x,\xi)|\cdot[\lambda(\xi)]^{-(m+\delta|\beta|-\rho|\alpha|)}\qquad(l=0,1,2\cdots).$$

In addition, we say a set $B\subset W_\lambda S^m_{\rho,\delta}$ is a bounded subset of $W_\lambda S^m_{\rho,\delta}$, if

$$\sup_{P\in B}\{|P|^{(m)}_l\}<+\infty\qquad\text{for every } l\in N_0.$$

Theorem 1 Suppose $P(x,\xi)\in W_\lambda S^m_{\rho,\delta}$, then the operator P defined by

$$(1)\qquad Pu(x)=\int e^{ix\xi}P(x,\xi)\hat{u}(\xi)d\xi\qquad u\in S(R^n_x)$$

is a linear continuous operator of S into S.

Proof Write $r(x,\xi)=e^{ix\xi}P(x,\xi)\hat{u}(\xi)$, we have

$$|r(x,\xi)|\leqslant C|P|^{(m)}_0|u|_{2(m_0m_++n+1)}\langle\xi\rangle^{-n-1}\in L_1(R^n_\xi)\qquad(m_+=\max\{0,m\})$$

which shows $Pu(x)$ is meaningful for $u\in S(R^n_x)$. In view of $\xi_jP(x,\xi)\in W_\lambda S^{m+1}_{\rho,\delta}$, $\partial_{x_j}P(x,\xi)\in W_\lambda S^{m+\delta}_{\rho,\delta}$, $\partial_{\xi_j}P(x,\xi)\in W_\lambda S^{m-\rho}_{\rho,\delta}$, it is easy to see that there is no fun-

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damental difference between $x^\tau\partial_x^\nu Pu(x)$ and $Pu(x)$. We can assert for any $\tau,\nu\in N_0^n$ constants $C_{\tau,\nu}$ and $l_{\tau,\nu}$ will be found, such that

$$|x^\tau\partial_x^\nu Pu(x)|\leqslant C_{\tau,\nu}|u|\,l_{\tau,\nu},$$

which means P is no other than a continuous operator of S into S.

Definition 2 A linear continuous operator P of S into S defined by (1) is called a pseudo-differential operator with a symbol $P(x,\xi)\in W_\lambda S^m_{\rho,\delta}$, which is denoted by $P(x,D_x)$. An operator set consisting of the whole of such operators will be denoted by $W_\lambda\mathfrak{S}^m_{\rho,\delta}$.

Definition 3 Assume that a function $a(y,\eta)$ is well defined on $R^{2n}_{y,\eta}$, $\chi(y,\eta)\in S(R^{2n}_{y,\eta})$, $\chi(0,0)=1$, if we have, for any of such $\chi(y,\eta)$,

$$\Big|\lim_{\varepsilon\to\infty}\int\int e^{-iy\eta}a(y,\eta)\chi(\varepsilon y,\varepsilon\eta)\,dyd\eta\Big|=C<+\infty,$$

then the limit C is called an oscillatory integral of the function $a(y,\eta)$ which is expressed as

$$Os\text{-}\int\int e^{-iy\cdot\eta}a(y,\eta)\,dyd\eta.$$

Definition 4 We say a function $P(x,\xi,x',\xi')\in C\ (R^{4n})$ is a symbol of the functional set $W_\lambda S^{m,m'}_{\rho,\delta}(-\infty<m,m'<+\infty,\ 0\leqslant\delta\leqslant\rho\leqslant1,\ \delta<1)$, if a constant $C_{\alpha,\alpha',\beta,\beta'}$ and a function $\lambda(\xi)$ can be found for any multi-indices $\alpha,\alpha',\beta,\beta'\in N_0^n$ such that

$$(2)\qquad |P^{(\alpha,\alpha')}_{(\beta,\beta')}(x,\xi,x',\xi')|\equiv|\partial_\xi^\alpha\partial_{\xi'}^{\alpha'}D_x^\beta D_{x'}^{\beta'}P|\leqslant C_{\alpha,\alpha',\beta,\beta'}[\lambda(\xi)]^{m+\delta(\beta)-\rho|\alpha|}\cdot[\lambda(\xi)+\lambda(\xi')]^{\delta|\beta|}\cdot[\lambda(\xi')]^{m'-\rho|\alpha'|},$$

where the function $\lambda(\xi)$ satisfies the condition (*) listed in definition 1.

For $P(x,\xi,x',\xi')\in W_\lambda S^{m,m'}_{\rho,\delta}$, we define its semi-norms by

$$|P|_l^{(m,m')}=\max_{|\alpha+\alpha'+\beta+\beta'|\leqslant l}\inf\{C_{\alpha,\alpha',\beta,\beta'}\}\qquad(l=0,1,2,\cdots)$$

in which $C_{\alpha,\alpha',\beta,\beta'}$ satisfes the inequality (2).

The following conditions are common to theorem 2-5: Suppose $P(x,\xi,x',\xi')\in W_\lambda S^{m,m'}_{\rho,\delta}$, $\alpha,\alpha',\beta,\beta'\in N_0^n$ arbitrarily, denote $\tau=m+m'+\delta|\beta+\beta'|-\rho|\alpha+\alpha'|$ and $q(x,\xi,x',\xi')=P^{(\alpha,\alpha')}_{(\beta,\beta')}(x,\xi,x',\xi')$, then we have

Theorem 2 A function $q_\theta(x,\xi)$ is well defined on $R^{2n}_{x,\xi}$ with $|\theta|\leqslant1$ by

$$q_\theta(x,\xi)=Os-\int\int e^{-iy\eta}q(x,\xi+\theta\eta,x+y,\xi)\,dyd\eta.$$

Proof Choose $\chi(y,\eta)\in S$, $\chi(0,0)=1$, put

$$r_{\theta,\varepsilon}(x,\xi,y,\eta)=e^{-iy\eta}q(x,\xi+\theta\eta,x+y,\xi)\cdot\chi(\varepsilon y,\varepsilon\eta),$$

$$(3)\qquad I_\varepsilon(x,\xi)=\int\int r_{\theta,\varepsilon}(x,\xi,y,\eta)\,dyd\eta.$$

Having observed that ξ, $|\theta|\leqslant1$, $|\varepsilon|<1$, $\varepsilon\neq0$ are fixed, we get

$$|r_{\theta,\varepsilon}|\leqslant C|\chi(\varepsilon y,\varepsilon\eta)|\langle\eta\rangle^{2m_+(m_++\delta|\beta+\beta'|)}\in L_1(R^{2n}_{y,\eta}).$$

Otherwise, by making use of the identical relations

$$e^{-iy\eta}=\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}e^{-iy\eta}=\langle y\rangle^{-2l}\langle D_\eta\rangle^{2l}e^{-iy\eta},$$

to integrate (3) by parts, we obtain

$$(4)\quad I_\varepsilon(x,\xi)=\int\int e^{-iy\eta}\langle y\rangle^{-2l}\langle D_\eta\rangle^{2l}\{\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}[q(x,\xi+\theta\eta,x+y,\xi)\chi(\varepsilon y,\varepsilon\eta)]\}dyd\eta.$$

Furthermore, we can verify the integrand in (4) belongs to $L_1(R^{2n}_{y,\eta})$, if and only if we take so large an l, then

(5) $$q_\theta(x,\xi)=\lim_{\varepsilon\to 0} I_\varepsilon(x,\xi)=$$
$$\iint e^{-iy\eta}\langle y\rangle^{-2l}\langle D_\eta\rangle^{2l}\{\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}[q(x,\xi+\theta\eta,x+y,\xi)]\}dyd\eta,$$

Theorem 3 $\partial_x^\nu\partial_\xi^\mu q_\theta(x,\xi)=Os-\iint e^{-iy\eta}\partial_x^\nu\partial_\xi^\mu q(x,\xi+\theta\eta,x+y,\xi)dyd\eta$, for $\nu,\mu\in N_0^m$.

Proof We only point out that we can differentiate (5) under the integral sign, so this theorem clearly holds true.

Theorem 4 There exist constants $C>O$ and $l\in N_0$ (both being independent of $|\theta|\leqslant 1$), such that

(6) $$|q_\theta(x,\xi)|\leqslant C|P|_l^{(m,m')}[\lambda(\xi)]^\tau.$$

Proof By making use of the identity $e^{-iy\eta}=(1+[\lambda(\xi)]^{2\delta}|y|^2)^{-l_0}(1+[\lambda(\xi)]^{2\delta}\cdot(-\Delta\eta))^{l_0}e^{-iy\eta}$ to integrate $q_\theta(x,\xi)$ by parts, we obtain

$$q_\theta(x,\xi)=\lim_{\varepsilon\to 0}\iint e^{-iy\eta}(1+[\lambda(\xi)]^{2\delta}|y|^2)^{-l_0}(1+[\lambda(\xi)]^{2\delta}(-\Delta\eta))^{l_0}[q(x,\xi+\theta\eta,x+y,\xi)\cdot\chi(\varepsilon y,\varepsilon\eta)]dyd\eta.$$

Denoting $r_\theta(x,\xi,y,\eta)=(1+[\lambda(\xi)]^{2\delta}|y|^2)^{-l_0}(1+[\lambda(\xi)]^{2\delta}(-\Delta\eta))^{l_0}q(x,\xi+\theta\eta,x+y,\xi)$, then $r_\theta(x,\xi,y,\eta)$ is absolutely integrable to y for $l_0>\frac{n}{2}$. Now. we divide the integral region into three parts: $\Omega_1\{\eta:|\eta|\leqslant[\lambda(\xi)]^\delta/2N\}$, $\Omega_2=\{\eta:[\lambda(\xi)]^\delta/2N\leqslant|\eta|\leqslant\lambda(\xi)/2N\}$ and $\Omega_3=\{\eta:|\eta|\geqslant\lambda(\xi)/2N\}$, writing $I_i(x,\xi)=\int_{\Omega_i}\left[\int e^{-iy\eta}r_\theta(x,\xi,y,\eta)dy\right]d\eta$, $(i=1,2,3)$, we have

(A) When $\eta\in\Omega_1$, it can be shown that

(7) $|I_1(x,\xi)|\leqslant\int_{\Omega_1}\left[\int(1+[\lambda(\xi)]^{2\delta}|y|^2)^{-l_0}\cdot\bar{C}_1|P|_{l_1}^{(m,m')}[\lambda(\xi)]^\tau dy\right]d\eta=C_1|P|_{l}^{(m,m')}[\lambda(\xi)]^\tau$, here C_1 is a constant, $l_1=2l_0+|\alpha+\alpha'+\beta+\beta'|$.

(B) We take an $l_2=2l_0+l_1$, then

(8) $|I_2(x,\xi)|\leqslant\bar{C}_2|P|_{l_2}^{(m,m')}[\lambda(\xi)]^{\tau+(2l_0-n)\delta}\cdot\int_{\Omega_2}|\eta|^{-2l_0}d\eta\leqslant C_2|P|_{l_2}^{(m,m')}[\lambda(\xi)]^\tau$.

(C) If we take so large an l that

$$\tau'+2l_0\delta-2l(1-\delta)<-n$$
$$m'+\tau'+2l_0\delta-2l(1-\delta)+(1-\delta)n<\tau$$

in which $\tau'=m_++\delta|\beta+\beta'|$, $l_3=l_1+2l$, then

(9) $|I_3(x,\xi)|\leqslant\int_{\Omega_3}|\eta|^{-2l}\left[\int|e^{-iy\eta}(-\Delta y)^l r_\theta(x,\xi,y,\eta)|dy\right]d\eta\leqslant C_3|P|_{l_3}^{(m,m')}[\lambda(\xi)]^\tau$.

Clearly, (7), (8) and (9) imply (6).

Theorem 5 $\{q_\theta(x,\xi)\}_{|\theta|\leqslant 1}$ is a bounded subset of the functional space $W_\lambda S^\tau_{\rho,\delta}$. Moreover, for every $l_0\in N_0$, there exist constants $C_0>0$ and $l_0'\in N_0$, which are independent of $|\theta|\leqslant 1$, such that

$$|q_\theta(x,\xi)|^{(r)}_{l_0} \leqslant C|P|^{(m,m')}_{l'_0}. \tag{10}$$

Proof Because of the linearity of the oscillatory integral, (10) follows from Theorem 3 and Theorem 4.

Theorem 6 Suppose the symbols of the pseudo-differential operators $P_1(x,D_x)$ and $P_2(x,D_x)$ are $P_1(x,\xi)\in W_\lambda S^{m_1}_{\rho,\delta}$, $P_2(x,\xi)\in W_\lambda S^{m_2}_{\rho,\delta}$ respectively, then the product of these two operators is also a pseudo-differential operator with a symbol $P(x,\xi)\in W_\lambda S^{m_1+m_2}_{\rho,\delta}$ defined by

$$P(x,\xi) = Os-\iint e^{-iy\eta}P_1(x,\xi+\eta)\cdot P_2(x+y,\xi)\,dyd\eta.$$

Proof It is obvious that $P(x,\xi)\in W_\lambda S^{m_1+m_2}_{\rho,\delta}$ and that, for any $u(x)\in S$, we have

$$P(x,D_x)u(x) = \int e^{ix\xi}[Os-\iint e^{-iy\eta}P_1(x,\xi+\eta)\cdot P_2(x+y,\xi)\,dyd\eta]\hat{u}(\xi)\,d\xi.$$

By applying the Lebesgue dominated convergence theorem, we can obtain

$$P(x,D_x)u(x) = \int e^{ix\eta}\left\{\int e^{-iy\eta}\left[\int e^{iy\xi}P_1(x,\eta)P_2(y,\xi)\hat{u}(\xi)\,d\xi\right]dy\right\}d\eta$$

$$= P_1(x,D_x)[P_2(x,D_x)u(x)] \qquad \text{for} \quad u\in S.$$

Theorem 7 Soppose $P(x,\xi)\in W_\lambda S^m_{\rho,\delta}$ is a symbol of a pseudo-differential operator $P(x,D_x)$, we define its conjugate operator P^* as $(Pu,v)=(u,P^*v)$ for $u,v\in S$, then P^* is also a pseudo-differential operator with a symbol $P^*(x,\xi)\in W_\lambda S^m_{\rho,\delta}$ defined dy

$$P^*(x,\xi) = Os-\iint e^{-iy\eta}\,\overline{P(x+y,\xi+\eta)}\,dyd\eta.$$

Proof There is no doubt that $P^*(x,\xi)\in W_\lambda S^m_{\rho,\delta}$. Furthermore, for any $u,v\in S$, we have

$$(u,\ P^*(x,D_x)v) = \int\left\{\int \overline{e^{ix\xi}\left[Os-\iint e^{-iy\eta}P(x+y,\xi+\eta)\,dyd\eta\right]\hat{v}(\xi)\,d\xi}\right\}u(x)\,dx$$

$$= \int\left\{\int e^{iy\eta}P(y,\eta)\hat{u}(\eta)\,d\eta\right\}\bar{v}(y)\,dy = (Pu,\ v).$$

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