第5卷 第2期
1985年4月

数学研究与评论
JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION

Vol.5 No.2
April 1985

A Generalization of Bellman-Gronwall Integral Inequality*

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Recently, Dhongade[1] and Yeh[2] studied some integral inequalities in R and R^n respectively, we find that their work can be improved. In this paper, we will improve their results, one of our results is best, i.e. it can not be improved again, another of our results is better than Yeh's[2].

Theorem 1 If

$$Q(x)\leqslant f(x)+g(x)\int_{x_0}^{x}h(s)Q(s)ds,\quad x_0\leqslant x\leqslant x_1 \tag{1}$$

where $g>0$, $h\geqslant 0$ and Q are integrable, f/g is absolutely continuous, then

$$Q(x)\leqslant g(x)\left[\frac{f(x_0)}{g(x_0)}\exp\left(\int_{x_0}^{x}h(s)g(s)ds\right)+\int_{x_0}^{x}\exp\left(\int_{s}^{x}g(t)h(t)dt\right)\left(\frac{f(s)}{g(s)}\right)'ds\right]. \tag{2}$$

Proof Dividing both members of inequality (1) by g, we have

$$\frac{Q(x)}{g(x)}\leqslant\frac{f(x)}{g(x)}+\int_{x_0}^{x}h(s)g(s)\ \frac{Q(s)}{g(s)}ds.$$

Using a well known inequality[3], we obtain

$$\frac{Q(x)}{g(x)}\leqslant\frac{f(x_0)}{g(x_0)}\exp\left(\int_{x_0}^{x}hgds\right)+\int_{x_0}^{x}\exp\left(\int_{s}^{x}hgdt\right)\left(\frac{f}{g}\right)'ds$$

i.e.

$$Q(x)\leqslant g(x)\left[\frac{f(x_0)}{g(x_0)}\exp\left(\int_{x_0}^{x}hgds\right)+\int_{x_0}^{x}\exp\left(\int_{s}^{x}hgdt\right)\left(\frac{f}{g}\right)'ds\right].$$

Remark 1 We compare the inequality (2) with Dhongade's inequality[1], at first, the later retrict f be nondecreasing, but we no, the second, Dhongade's inequality is not best, but inequality (2) not only is better than Dhongade's inequality but also is unimproved again.

*Received May 18, 1983.

Example 1 If

$$Q(x) \leqslant e^{x} + e^{\frac{1}{2}} \int_{0}^{x} e^{-\frac{s}{2}} Q(s)\, ds, \tag{3}$$

according to [1], we have

$$Q(x) \leqslant e^{\frac{5}{2}x}, \tag{4}$$

but according to (2), we have

$$Q(x) \leqslant 2e^{\frac{3}{2}x} - e^{x}, \tag{5}$$

the formula (5) is accurater than (4).

Theorem 2 If

$$Q(x) \leqslant f(x) + \sum_{i=1}^{n} g_i(x) \int_{x_0}^{x} h_i(s) Q(s)\, ds, \qquad x_0 \leqslant x \leqslant x_1 \tag{6}$$

where $g_i > 0$, $h \geqslant 0$ and Q are integrable, $i = 1, 2, \cdots, n$. $E^i f / g$ is absolutely continuous, $i = 1, 2, \cdots, n$, then

$$Q(x) \leqslant E^n f, \tag{7}$$

where E^h is defined as following

$$E^0 f = f,$$

$$E^k f = g_k(x) \left[\frac{f(x_0)}{g_k(x_0)} \exp\left(\int_{x_0}^{x} g_k(s) h_k(s)\, ds \right) + \right.$$

$$\left. + \int_{x_0}^{x} \exp\left(\int_{s}^{x} g_k(t) h_k(t)\, dt \right) \left(\frac{E^{k-1} f(s)}{g_k(s)} \right)' ds \right], x_0 \leqslant x < x_1, \quad k = 1, 2, \cdots, n. \tag{8}$$

Proof Using mathematical induction.

It is obvious, inequality (7) holds if $n = 1$; Assume that inequality (7) holds for $n = k$, so we have

$$Q(x) \leqslant E^k f \leqslant E^k f + g_{k+1} \int_{x_0}^{x} h_{k+1}(s) Q(s)\, ds.$$

Using theorem 1 and noticing $E^k f(x_0) = f(x_0)$, we have

$$Q(x) \leqslant g_{k+1}(x) \left[\frac{f(x_0)}{g_{k+1}(x_0)} \exp \int_{x_0}^{x} g_{k+1}(s) h_{k+1}(s)\, ds \right.$$

$$\left. + \int_{x_0}^{x} \exp\left(\int_{x_0}^{r} g_{k+1}(s) h_{k+1}(s)\, ds \right) \left(\frac{E^k f}{g_{k+1}} \right)' ds \right] = E^{k+1} f.$$

The proof is completed.

Dhondage obtained the following inequality

$$Q(x) \leqslant E^n f \tag{9}$$

where

$$E^0 f = f,$$

$$E^k f = f(x) E^{k-1} g_k(x) \exp\left(\int_0^x h_k E^{k-1} g_k ds\right), \quad k = 1, 2, \cdots, n. \tag{10}$$

Remark 1 also holds for theorem 2.

Example 2 If

$$Q(x) \leqslant x^3 + \int_0^x (1+s) Q(s) ds + e^x \int_0^x e^{-\frac{s^2}{2}} Q(s) ds, \qquad 0 < x < \infty.$$

According to (9), we have

$$Q(x) \leqslant E^2 f = x^3 \exp\left(\frac{4x+x}{2}\right) \exp\left(\frac{e^{2x}}{2} - 1\right). \tag{11}$$

According to (7), we have

$$Q(x) \leqslant x^3 e^{\frac{x^2}{2}} + \frac{31 e^{x e^{\frac{1}{2}}}/2}{(e^{\frac{1}{2}}-1)^4} + \frac{x^3 e^x}{e^{\frac{1}{2}}-1} - \frac{3x^2 e^x}{(e^{\frac{1}{2}}-1)^2} - \frac{3!\, x e^x}{(e^{\frac{1}{2}}-1)^3} - \frac{3!\, e^x}{(e^{\frac{1}{2}}-1)^4}. \tag{12}$$

It is obvious, inequality (12) is accurater than (11), especially, as $x > 1$, the difference between (11) and (12) is very large.

Yeh[2] studied integral inequality in R^n. We will prove the following theorem before improving Yeh's work.

Theorem 3 If

$$Q(x) \leqslant f(x) + g(x) \int_{x_0}^x h(s) Q(s) ds \quad x_0 \leqslant x \leqslant x_1, \tag{13}$$

where $f > 0$, $Q \geqslant 0$, $h \geqslant 0$ and $g \geqslant 0$ are continueus and $\frac{g}{f}$ is nonincreasing, then

$$Q(x) \leqslant f(x) \exp\left(\int_x^x h(s) g(s) ds\right). \tag{14}$$

Proof From (13) and noticing g/f is nonincreasing

$$\frac{Q}{f} \leqslant 1 + \frac{g}{f} \int_{x_0}^x h(s) Q(s) ds \leqslant 1 + \int_{x_0}^x h(s)\, g(s) \frac{Q(s)}{f(s)} ds,$$

so

$$\frac{Q}{f} \leqslant f \exp \int_{x_0}^x g h ds,$$

thus

$$Q \leqslant f \exp\left(\int_{x_0}^x h(s) g(s) ds\right).$$

Inequality (14) is better than Dhongade's inequality. In fact, we see example

1, according to (14), we have

$$Q(x) \leqslant e^{2x}, \tag{15}$$

it is accurater than inequality (4).

Of course, inequality (14) is not better than (2), but inequality of this form is convinient for extending to n dimension.

Now we consider inequality of n dimension function.

Let $x=(x,\cdots,x)$ is a n dimension vector, we define

$$\int_{x^0}^{x} \cdots ds = \int_{x_1^0}^{x_1} \cdots \int_{x_n^0}^{x_n} \cdots ds_n \cdots ds_1 \tag{16}$$

$$D_i = \frac{\partial}{\partial x_i} \qquad i=1,2,\cdots,n.$$

Inequality $x^0 \leqslant x$ implies if and only if $x_i^0 \leqslant x_i$, $i=1,2,\cdots,n$.

Theorem 4 If

$$w(x) \leqslant f(x) + g(x)\int_{x^0}^{x} h(s)w(s)ds, \qquad x^0 \leqslant x \leqslant x^1 \tag{17}$$

where all functions are continuous in R^n, $f(x)>0$ and $\frac{g}{f}$ is nonnegative and non-increasing for every x_i, $i=1,2,\cdots,n$, $w(x)\geqslant 0$, $h(x)\geqslant 0$, then

$$w(x) \leqslant f(x)\exp\left(\int_{x^0}^{x} h(s)g(s)ds\right), \tag{18}$$

i. e. inequality (13) also holds in R^n.

Proof From (17), we have

$$\frac{w(x)}{f(x)} \leqslant 1 + \int_{x^0}^{x} h(s)g(s)\frac{w(s)}{f(s)}ds, \qquad x^0 \leqslant x \leqslant x^1.$$

Defining $r(x) = 1 + \int_{x^0}^{x} h(s)g(s)\frac{w(s)}{f(s)}ds$. Obviously, $r(x_0)=1$ and $D_1\cdots D_n r(x) = h(x)g(x)\frac{w(x)}{f(x)}$, so $\frac{D_1\cdots D_n r(x)}{r(x)} \leqslant h(x)g(x)$, thus

$$\frac{r(x)D_1\cdots D_n r(x)}{r(x)} \leqslant h(x)g(x) + \frac{D_n r(x)D_1\cdots D_{n-1}r(x)}{r^2(x)}.$$

Because $D_n r(x)D_1\cdots D_{n-1}r(x) \geqslant 0$, so that

$$D_n\left(\frac{D_1\cdots D_{n-1}r(x)}{r(x)}\right) \leqslant h(x)g(x).$$

Integrating both sides from x_n^0 to x_n, we have

$$\frac{D_1\cdots D_{n-1}r(x)}{r(x)}-\frac{D_1\cdots D_{n-1}r(x_1,\cdots,x_{n-1},x_n^0)}{r(x\cdots,x,x)}\leqslant\int_{x_n^0}^{x_n}h(x_1,\cdots x_{n-1},s_n)g(x_1,\cdots,x_{n-1},\ s_n)ds_n,$$

but $D_1\cdots D_{n-1}r(x_1,\cdots,x_{n-1},x_n^0)=0$, so

$$\frac{D_1\cdots D_{n-1}r(x)}{r(x)}\leqslant\int_{x_n^0}^{x_n}h(x_1,\cdots,x_{n-1},s_n)g(x_1,\cdots,x_{n-1},s_n)ds_n.$$

Repeating above procedure, we obtain

$$\frac{D_1r(x)}{r(x)}\leqslant\int_{x_2^0}^{x_2}\cdots\int_{x_n^0}^{x_n}h(x_1,s_2,\cdots,s_n)g(x_1,s_2,\cdots,s_n)ds_n\cdots ds_2.$$

Integrating for x_1, we obtain $\ln r\ (x)\leqslant\int_{x_0}^{x}h(s)g(s)ds$, i.e. $r(x)\leqslant\exp\int_{x_0}^{x}h(s)g(s)ds$, so

$$w(x)\leqslant f(x)\exp\int_{x_0}^{x}h(s)g(s)ds.$$

Similarly, we can obtain the following theorem.

Theorem 5 If

$$w(x)\leqslant f(x)+\sum_{i=1}^{m}g_i(x)\int_{x^0}^{x}h_i(s)w(s)ds,\ x^0\leqslant x\leqslant x^1 \tag{18}$$

where all functions are continuous in R^n, $f(x)>0$, $g_i/E^if\geqslant0$ $i=1,2,\cdots n$ is non-increasing, $w(x)\geqslant0$, $h_i(x)\geqslant0$, $i=1,2,\cdots n$. Then

$$w(x)\leqslant E^mf(x),$$

where

$$E^0f(x)=f(x),\ E^kf(x)=(E^{k-1}f(x))\exp\int_{x^0}^{x}h_k(s)g_k(s)ds,\quad k=1,2,\cdots,m, \tag{19}$$

$$x^0\leqslant x\leqslant x^1.$$

Remark 2 In the paper [2], Yeh obtained

$$w(x)\leqslant f(x)g(x)\exp\left(\int_0^x h(s)g(s)ds\right), \tag{20}$$

under light different hypotheses, where $g(x)\geqslant1$.

We compare the inequality (18) with Yeh's (20). It is obvious, inequality (18) is better than (20).

Similarly, inequality (19) also improves Yeh's corresponding inequality in [1].

Example 3 [2] In R_+^2, we consider the inequality

$$w(x_1,x_2)\leqslant x_2^2+x_2^3+\int_0^{x_1}\int_0^{x_2}w(s_1,s_2)ds_1ds_2+e^{x_1x_2}\int_0^{x_1}\int_0^{x_2}e^{s_1-s_1s_2}w(s_1,s_2)ds_1ds_2,$$

where $f(x,x)=x_1^2+x_2^3$, $g_1(x,x)=1$, $g_2(x,x)=e^{x_1x_2}$, $h_1(s,s)=1$ and $h_2(s,s)=e^{s_1-2s_1s_2}$.

From our result (19), we have

$$w(x)\leqslant E^2f(x)\leqslant(x_1^2+x_2^3)\exp(x_1x_2)\exp(x_2e^{x_1}-x_2).$$

From Yeh's result[2], we have

$$w(x)\leqslant(x_1^2+x_2^3)\exp(2x_1x_2)\exp(x_2e^{x_1}-x_2).$$

Obviously, the former is accurater than the later.

References

[1] Dhongade, U. D. and Deo, S. G., Pointwise estimates of solutions of some Volterra integral equations, *J. Math. Anal.Appl.*, 45 (1974), 615—628.

[2] Cheh-Chihyeh, Bellman-Bihari integral inequalities in several independent variables, *J.Math Anal. Appl.*,87 (1982), 311—321.

[3] Sansone, G. and Conti, R., Non-linear Differential Equations, Pergamon Press, Oxford 1964.

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答丁协平同志

颜心力

在本刊1982年第4期上见到丁协平同志指出拙作(简称文1)《不动点定理》是他人“结果(简称文2)的改述，其证明也是类似的”.并说“这一方法主要用于证明下述事实：若算子$T:X\to X$的某次迭代满足具有唯一不动点性质的压缩条件，则T^n的不动点也是T的不动点”.最后进一步说文1“只作了转化工作而未给出寻找F的新方法，因此无法认为该文是有意义的”.

事实上，只要较为仔细地分析一下，就给发现二者至少有两处本质差异：1°文1设T，F映距离空间到自身；而文2要求T，F映Banach空间的闭集到自身、2°文1设TF有唯一不动点，而文2则要求F压缩．第1°条本质差异非常明显，可不论．对第2°条，众所周知，压缩、拓扑度、Schauder不动点定理、临界点理论以及为数众多的不动点定理(有些还需附加一定条件)均可成为算子具有唯一不动点的充分条件(压缩仅其中之一)．不妨看一个简单例子．设$X=(0,1]$，$Tx=x^\alpha$，$Fx=x^\beta$，α,β为互异的大于2的实数.显然，T，F可换，TF非压缩且非T^n，但它在X有唯一不动点1．此例显然适合文1的条件而不为文2与丁的评论所满足．故改述二字不实．至于证明无需借鉴是不言自明的．

要求给出寻找F的新方法，等价于要求判定一个任意算子T的不动点的存在性，显然是人们无法解决的一个问题．