第5卷 第2期
1985年4月

数 学 研 究 与 评 论
JOURNAL OF MATHEMATICAL
RESEARCH AND EXPOSITION

Vol.5 No.2
April 1985

On a Problem of Approximation by Linear Means*

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1. Introduction Let $f\in C_{2\pi}$ and

$$f(x)\sim\frac{1}{2}a_0+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx) \tag{1.1}$$

be its Fourier series. We denote by $s_n(x)=s_n(f,x)$ the nth partial sum of series (1.1). Let $\{p_n\}$ be a sequence of positive constants such that

$$P_n=p_0+p_1+\cdots+p_n\to\infty \quad as\ n\to\infty$$

and

$$0<np_n\leqslant cP_n^{**} \text{ for } n=1,2,\ldots;\ p_0>0.$$

It is well known that the Nörlund means of series (1.1) are

$$N_n(f,x)=\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}s_k(x).$$

On the degree of approximation of $f(x)$ by the Nörlund means of the Fourier series (1.1), A. S. B. Holland, B. N. Sahney and J. Tzimbalario[1] proved the following

Theorem A The following estimate

$$\|f(x)-N_n(f,x)\|=O\left(\frac{1}{P_n}\sum_{k=1}^{n}\frac{P_k\omega\left(f,\frac{1}{k}\right)}{k}\right) \tag{1.2}$$

is valid, where $\omega(f,\delta)$ is the modulus of continuity of f. Furthermore they asked that:

(1.3) If Theorem A can be extended to matrix summability?

Later, P. D. Kathal, A. S. B. Holland and B. N. Sahney[2] partially answered this problem. They proved the following

Theorem B If $\{A_{n,k}\}$ is monotonic non-increasing for all $0\leqslant k\leqslant n$ and $A_{n,k}\geqslant 0$ $(0\leqslant k\leqslant n)$, $A_{n,k}=0$ $(k>n)$, then we have

* Received Mar. 17, 1983.

** In this paper, c is a positive constant, not the same at each occurrence.

$$\|f(x)-T_n(f,x)\|=O\left\{\sum_{k=1}^{n}\frac{D_{n,k}\omega\left(f,\frac{1}{k}\right)}{k}\right\},$$

where

$$T_n(f,x)=\sum_{k=0}^{n}\Lambda_{n,k}s_k(x) \text{ and } D_{n,k}=\sum_{r=0}^{k}\Lambda_{n,r}.$$

In this paper, we shall give fuller answer of the problem (1.3). From the obtained result it follows that under the conditions of Theorem B the estimate

$$\sum_{k=0}^{n}\Lambda_{n,k}|f(x)-s_k(x)|=O\{\sum_{k=0}^{n}\Lambda_{n,k}E_k(f)\} \tag{1.4}$$

is valid, where $E_k(f)$ denotes the best approximation of $f(x)$ by trigonometric polynomials of order at most n.

Applying our result to the approximation of the Nörlund means, we have

$$\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}|f(x)-s_k(x)|=O\left(\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}E_k(f)\right),$$

where $\{p_k\}$ is monotonic increasing.

2. Main results We shall need the following known result (see [3]).

Lemma 2.1 Let $p>0$, $v_n=O(n)$ $(n\in N)$, then

$$\left\{\frac{1}{n}\sum_{k=v_n}^{v_n+n-1}|f(x)-s_k(x)|^p\right\}^{\frac{1}{p}}=O(E_{v_n}(f)), \tag{2.1}$$

where sign "O" is independent of n, x and f.

Now, we prove the following

Theorem 2.2 Let $p>0$ and $\{\Lambda_{n,k}\}$ satisfy the conditions: (i) $\{\Lambda_{n,k}\}$ is monotonic non-increasing for all $0\leqslant k\leqslant n$ or (ii) there exist two positive constants a and b $(a<b)$ such that

$$a\leqslant\Lambda_{n,v}/\Lambda_{n,2k}\leqslant b \quad (k=0,1,\cdots,n;\ v=k,\cdots,2k),$$

then

$$\left\{\sum_{k=0}^{n}\Lambda_{n,k}|f(x)-s_k(x)|^p\right\}^{\frac{1}{p}}=O\left\{\left(\sum_{k=0}^{n}\Lambda_{n,k}E_k^p(f)\right)^{\frac{1}{p}}\right\},$$

where the sign "O" is independent of n, x and f.

Proof First let $\{\Lambda_{n,k}\}$ satisfy the condition (i) and m be a positive integer such that $2^{m-1}\leqslant n<2^m$, then we have

$$A_n^p(x)=\left\{\sum_{k=0}^{n}\Lambda_{n,k}|f(x)-s_k(x)|^p\right\}^{\frac{1}{p}}\leqslant c(p)\left\{\left(\Lambda_{n,0}|f(x)-s_0(x)|^p\right)^{\frac{1}{p}}\right.$$

$$\left.+\left(\sum_{k=1}^{m-1}\sum_{v=2^{k-1}}^{2^k-1}\Lambda_{n,v}|f(x)-s_v(x)|^p\right)^{\frac{1}{p}}+\left(\sum_{v=2^{m-1}}^{n}\Lambda_{n,v}|f(x)-s_v(x)|^p\right)^{\frac{1}{p}}\right\}.$$

Observing the fact $|f(x)-s_0(x)|\leqslant 2E_0(f)$ and the monotonicity of $\{\Lambda_{n,k}\}$ and using (2.1), we get

$$A_n^p(x)\leqslant c(p)\left\{\left(2^p\Lambda_{n,0}E_0^p(f)\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{m-1}\Lambda_{n,2^{k-1}}\cdot 2^{k-1}E_{2^{k-1}}^p(f)\right)^{\frac{1}{p}}+\left(\Lambda_{n,2^{m-1}}\cdot 2^{m-1}E_{2^{m-1}}^p(f)\right)^{\frac{1}{p}}\right\}$$

$$\leqslant c(p)\left\{\left(\Lambda_{n,0}E_0^p(f)\right)^{\frac{1}{p}}+\left(\Lambda_{n,1}E_1^p(f)\right)^{\frac{1}{p}}+\left(2\sum_{k=2}^{m}\Lambda_{n,2^{k-1}}\sum_{v=2^{k-2}+1}^{2^{k-1}}E_v^p(f)\right)^{\frac{1}{p}}\right\}$$

$$\leqslant c(p)\left\{\left(\Lambda_{n,0}E_0^p(f)\right)^{\frac{1}{p}}+\left(\Lambda_{n,1}E_1^p(f)\right)^{\frac{1}{p}}+\left(\sum_{k=2}^{m}\sum_{v=2^{k-2}+1}^{2^{k-1}}\Lambda_{n,v}E_v^p(f)\right)^{\frac{1}{p}}\right\}$$

$$=O\left\{\left(\sum_{k=0}^{n}\Lambda_{n,k}E_k^p(f)\right)^{\frac{1}{p}}\right\}.$$

If $\{\Lambda_{n,k}\}$ satisfies the condition (ii), the proof is analogous. (see the proof of the following Theorem 2.5) The proof is complete.

It is clear that Theorem 2.2 implies Theorem B. We have the following

Corollary 2.3 If $\{\Lambda_{n,k}\}$ is monotonic non-increasing for all $0\leqslant k\leqslant n$, then

$$\|f(x)-T_n(x)\|\leqslant\left\|\sum_{k=0}^{n}\Lambda_{n,k}|f(x)-s_k(x)|\right\|=O\left(\sum_{k=1}^{n}\frac{D_{n,k}E_k(f)}{k}\right).$$

Corollary 2.4 Let $p>0$ and $\{p_n\}$ be monotonic increasing, then

$$\left\|\sum_{k=0}^{n}\frac{p_{n-k}}{P_n}|f(x)-s_k(x)|^p\right\|=O\left(\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}E_k^p(f)\right).$$

If $\{\Lambda_{n,k}\}$ does not satisfy the conditions of theorem 2.2, we have the following

Theorem 2.5 Let $\{\Lambda_{n,k}\}$ satisfy the following conditions: (1) there exist two positive constans a and $b(a<b)$ such that

$$a\leqslant\Lambda_{n,v}/\Lambda_{n,k}\leqslant b \text{ for } k\leqslant v\leqslant 2k,\ 0\leqslant k\leqslant n/4;$$

(2) there is a $r>1$ such that

$$\left\{\frac{1}{n}\sum_{k=[n/2]}^{n}(\Lambda_{n,k})^r\right\}^{\frac{1}{r}}=O\left(\frac{1}{n}\right),$$

then we have

$$\left\{\sum_{k=0}^{n}\Lambda_{n,k}|f(x)-s_k(x)|^p\right\}^{\frac{1}{p}}=O\left\{\left(E_{[\frac{n}{2}]}^p(f)+\sum_{k=0}^{[\frac{n}{2}]}\Lambda_{n,k}E_k^p(f)\right)^{\frac{1}{p}}\right\},$$

where sign "O" is independent of n,x and f.

Proof Let $l=[n/2]$ and m_0 be a positive integer such that $2^{m_0-1}\leqslant l<2^{m_0}$, then

$$(A_n^p(x))^p=\sum_{k=0}^{n}\Lambda_{n,k}|f(x)-s_k(x)|^p=\sum_{k=1}^{l}+\sum_{k=l+1}^{n}=\sum\nolimits_1+\sum\nolimits_1\ \text{(say)}. \tag{2.2}$$

Using the condition (1), we have

$$\sum\nolimits_1=\Lambda_{n,0}|f(x)-s_0(x)|^p+$$
$$+\sum_{k=1}^{m_0-1}\sum_{v=2^{k-1}}^{2^k-1}\Lambda_{n,k}|f(x)-s_v(x)|^p+\sum_{v=2^{m_0-1}}^{l}\Lambda_{n,v}|f(x)-s_v(x)|^p$$
$$\leqslant O\left\{\Lambda_{n,0}E_0^p(f)+\sum_{k=1}^{m_0-1}b\Lambda_{n,2^{k-1}}\cdot 2^{k-1}E_{2^{k-1}}^p(f)+b\Lambda_{n,2^{m_0-1}}2^{m_0-1}E_{2^{m_0-2}}^p(f)\right\}$$
$$=O\left\{\Lambda_{n,0}E_0^p(f)+\Lambda_{n,1}E_1^p(f)+\sum_{k=2}^{m_0}\Lambda_{n,2^{k-1}}\sum_{v=2^{k-2}+1}^{2^{k-1}}E_v^p(f)\right\} \tag{2.3}$$
$$=O\left(\sum_{k=0}^{l}\Lambda_{n,k}E_k^p(f)\right).$$

Now, we estimate $\sum_2$. From the condition (2) it follows

$$n^{\frac{1}{r'}}\Big\{\sum_{k=[n/2]}^{n}\Lambda_{n,k}^{r}\Big\}^{\frac{1}{r}}=O(1),\quad \frac{1}{r}+\frac{1}{r'}=1. \tag{2.4}$$

Using (2.1), (2.4) and Hölder inequality, we get

$$\begin{aligned}\sum\nolimits_2&\leqslant\sum_{k=[n/2]}^{n}\Lambda_{n,k}|f(x)-s_k(x)|^{p}\\&\leqslant\Big\{\sum_{k=[n/2]}^{n}\Lambda_{n,k}^{r}\Big\}^{\frac{1}{r}}\Big\{\sum_{k=[n/2]}^{n}|f(x)-s_k(x)|^{pr'}\Big\}^{\frac{1}{r'}}\\&=O(E_{[n/2]}^{p}(f))\cdot n^{\frac{1}{r'}}\Big(\sum_{k=[n/2]}^{n}\Lambda_{n,k}^{r}\Big)^{\frac{1}{r}}=O(E_{[n/2]}^{p}(f)).\end{aligned} \tag{2.5}$$

Finally, combining (2.2), (2.3) and (2.5), Theorem 2.5 follows.

Corollary 2.6 If $\{p_k\}$ satisfies the following conditions:

(i) there exist two positive constants a and b $(a<b)$ such that

$$a\leqslant p_{n-v}/p_{n-k}\leqslant b \quad \text{for } k\leqslant v\leqslant 2k,\ 0\leqslant k\leqslant n/4;$$

(ii) there is a $r>1$ such that

$$\Big\{\frac{1}{n}\sum_{k=0}^{[n/2]}p_k^{r}\Big\}^{\frac{1}{r}}=O\Big(\frac{P_n}{n}\Big),$$

then for $p>0$,

$$\Big\{\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}|f(x)-s_k(x)|^{p}\Big\}^{\frac{1}{p}}=O\Big\{\Big(E_{[n/2]}^{p}(f)+\sum_{k=0}^{[n/2]}\frac{p_{n-k}}{P_n}E_k^{p}(f)\Big)^{\frac{1}{p}}\Big\}.$$

It is clear that in some cases, for example, in the case the Nörlund means with $p=1/(k+1)$ $(k=0,1,\cdots,n)$, condition (2) of Theorem 2.5 is not adequate. In this case, similarly, we have

Theorem 2.7 Let $\{\Lambda_{n,k}\}$ satisfy the condition: there exist two positive constants a and b $(a<b)$ such that

$$a\leqslant\Lambda_{n,v}/\Lambda_{n,k}\leqslant b\quad (k\leqslant v\leqslant 2k;\ 0\leqslant k\leqslant n/4),$$

then

$$\Big\{\sum_{k=0}^{n}\Lambda_{n,k}|f(x)-s_k(x)|^{p}\Big\}^{\frac{1}{p}}=O\Big\{\Big(\ln^{p}nE_{[n/2]}^{p}(f)+\sum_{k=0}^{[n/2]}\Lambda_{n,k}E_k^{p}(f)\Big)^{\frac{1}{p}}\Big\}.$$

Corollary 2.8 Let $p>0$, then the following estimate is valid:

$$\begin{aligned}&\Big\{\frac{1}{H_n}\sum_{k=0}^{n}\frac{1}{n-k+1}|f(x)-s_k(x)|^{p}\Big\}^{\frac{1}{p}}\\&=O\Big\{\Big(E_{[n/2]}^{p}(f)\ln^{p}n+\frac{1}{n\ln n}\sum_{k=0}^{[n/2]}E_k^{p}(f)\Big)^{\frac{1}{p}}\Big\},\end{aligned}$$

where $H=1+1/2+\cdots 1/(n+1)$.

References

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[2] Kathal, P.D., Holland, A.S.B. and Sahney, B.N., A Class of Continuous Functions and their Degree of Approximation, *Acta Math. Acad. Sci. Math. Hungar.*, 30(1977), 227—231.

[3] 谢庭藩，多项式逼近函数的几个问题，数学进展，13 (1984), 23—36.