第5卷 第2期
1985年4月

数 学 研 究 与 评 论
JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION

Vol.5 No.2
April 1985

A Probability Anecdote

K. L. Chung(钟开莱)

The use of probability ideas to solve apparently unrelated problems is a fascinating subject. Paul Erdős is a noted exponent of such uses and has proved many results in analysis, number theory and combinatorics in this way. A famous example is Serge Bernstein's brilliant proof of Weierstrass's theorem on approximation of continuous functions by polynomials. Another example is the nondifferentiability of Brownian sample functions (of which a quick proof is due to Erdős) which implies at once Weierstrass's fumous result (cf. Titchmarsh, Theory of Functions, second edition § 11.22). Recently, I encountered an amusing example in a qualifying examination given last summer at Stanford. It is the first problem in Part 2 of Real analysis, and runs as follows.

Problem. Lct q be a measurable nonpositive function in $[0, \infty)$. Provc that there is a unique solution φ of thc equation

$$\varphi''(x) + q(x)\varphi(x) = 0 \tag{1}$$

that is nonnegative and nonincreasing in $[0, \infty)$, with $\varphi(0) = 1$.

My interest was aroused because I saw a probability solution at once. In fact it is given, slightly implicitly, in my article in this journal, Vol. 2, No.3 (1982), Section 6, more specifically Proposition 2 there. Instead of the finite interval considered there, we can treat the half-line in this problem because $q \leqslant 0$. Here is the solution:

$$\varphi(x) = E^{x'}\left\{\exp\left[\int_0^{T_x} 2q(x_t)\,dt\right]\right\}, \qquad x \in [0, \infty) \tag{2}$$

where $\{X(t),\ t \geqslant 0\}$ is the Brownian motion on line, E^X the mathematical expectation starting from x, and T_0 the hiting time of $\{0\}$. It is manifest that $0 \leqslant \varphi \leqslant 1$. Note that φ is the solution of (1) give in my article then goes through without change. To see monotonicity let $x < x'$, then it follows from (2) and the strong Markov property that

$$\varphi(x') = E^{x'}\left\{\mathrm{exq}\left[\int_0^{T_x} 2q(x_t)\,dt\right]\right\}\cdot\varphi(x). \tag{3}$$

Hence $\varphi(x') \leqslant \varphi(x)$ because the first factor on the right side of (3) is between 0 and 1. This is the end of the proof of the existence part of the problem. The uniqueness part is quite easy by elementary analysis and left to the reader (see below).

In my cited article in this journal I discussed a more difficult case when q is assumed only to be bounded. Then even for a bounded domain of x the corresponding expectation in (2) may be infinite. Hence an additional assumption must be made to ensure a solution, as discussed there. The result has been extended to any dimension and to a class of unbounded q, but that is not easy. Zhao Zhongxin (赵忠信) and I are planning a monograph on the subject.

To return to the examination problem, I was curious to see how it can be solved "analytically". [Some probabilists, notably Doob, objected to the word because probability analysis is also analysis!] of the nine students who took this problem (there was some choice), all except one did next to nothing. Only one had the correct idea of a solution, but did not supply enough details to justify his claims. I gave him six points out of ten. To write down a complete solution to satisfy myself, it took me about three handwritten sheets. Thinking that I might have missed an easier way, I asked my doctorate student Liao Ming (廖明) (already teaching in the University of Florida by that time) to spell out a proof independently. His first proof covered more than one typed page, but omitted many details. At my request he supplied these in two more pages. I was then convinced that there was no quick solution by elementary analysis, unless one does a lot of "handwaving" (指手画脚胡说乱道). The latter is a bad habit often indulged in by experts, and must be avoided by beginners. I told the young man in charge of that exmination of my findings. He now seems to agree with me that while the problem is a good one, it may not be suitable for an examination. We do not know who originally made up the problem. My memory is that it is a very special case of "oscillation properties" of solutious of linear differential equations. There is a well known paper by H. Weyl (1910) on the subject. The problem may have appeared in some textbook. I do not know, nor does the young man.

Let me indicate the analytic solution. We are given $\varphi(0)$: put $\varphi'(0)=b$. By the initial value theorem for ODE, there is a unique solution with these data. If $b\geqslant 0$, then φ is nondecreasing. If $b<0$ and $|b|$ is large enough, φ will have a zero and become negative after. For each $a>0$, there exists a unique solution such that $\varphi(a)=0$. [This step is not trivial.] Call this solution f_a, then $f_a'<0$ everywhere. Let $c=\sup f_a'(0)$ where the sup is over all $a>0$. Then the solution with initial values $\varphi(0)=1$ and $\varphi'(0)=c$ is the desired solution. The proof depends on the comparison of solutions with same initial value and different initial derivatives, and on the theorem about the continuous variation of solutions with the initial data, carefully applied. Readers who have studied ODE should be able to work out the details with these hints. The sketch is given here to show what fundamentally different ideas are used in this classic method. Yet it only yields an existence proof: indeed it seems hard to determine the critical value c above. In contrast, the probability solution is constructive: it is explicitly exhibited in (2) for all (who understand the formula) to see. This is an important feature in many applications of probability methods. S. Bernstein not only proved Weierstrass's old theorem but introduced a brand new way of constructing the approximating polynomials which now bear his name. Later authors including myself have exploited his idea to give "transparent" proofs of other approximation formulas: see the article by D. Pfeifer in Vol. 2, No. 4 of this journal. The explicit formula for the solution of (1) given in (2) illustrates the point. The required knowledge of Brownian motion is elementary enough that I gave the material for the cited article in a course for upper class undergraduates and beginning graduates. Students of mathematics should be encouraged to study such subjects as part of their basic education. Historically, the introduction of Brownian motion as a mathematical tool can be attributed to Albert Einstein and Nobert Wiener. It furnishes an infusion of new blood into old physics and old analysis.(有志之士，盍兴乎来!)