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A Survey of Numerical Method for Solitary Waves(I)*

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Abstract

This paper is devoted to introduce the history of numerical study of solitons and the main results for K. D. V. equation, R. L. W. equation, Klein-Gordon equation, Sine-Gordon equation and so on.

The technique for constructing difference schemes based on conservation and characteristic is discussed. The two main methods for analysis of stability is given too. Then several finite element methods are discussed. The final part of this paper is for various spectral methods.

I Introduction

Russell discovered solitary wave in 1834 (See Russell, 1844). Korteweg de Vries (1895) studied such problem again. After them people did not fix the attention on it for a long time. Since Zabusky and Kruskal (1965) solved K.D. V. equation numericaly and discovered some properties of solitons, many scientists have worked in this field. There are a lot of literatures for solitons. such as Jeffrey & Kakutani (1972), Scott, Chu & Mclaughlin (1973), Whitham (1974), Lax (1976), Miura (1976), Strauss (1978), Makhankov (1978), Lamb (1980) and Bullough & Caudrey (1980). On the other hand, numerical study has developed rapidly, which could be devided into three parts. (A) Finite difference methods. The first scheme for K. D. V. equation is the second order accurate leap-frog scheme by Zabusky & Kruskal (1965). Dissipative schemes were considered by Vliegenthart (1971) and Hopscotch technique by Greig & Morris (1976). Kuo Pen-yu (1976) proposed several schemes based on conservation. Peregrine (1966), Eilbeck & McGuire (1977) and Wu Hua-mo & Kuo Pen-yu (1983) solved R. L. W. equation. For nonlinear Klein-Gordon equation, there are conservative schemes by Strauss & Vazquéz (1978) and others. Perring & Skyrme (1962) gave the first scheme to solve Sine-Gordon equation and the others followed.
(B) Finite element methods. Wahlbin (1974) employed dissipative finite element scheme for K. D. V- equation, implemented by Alexander & Morris (1979). Winther (1981) proposed another scheme. Sanz-Serna & Christie (1981) considered Petrov-Galerkin method. More recently Mitchell & Schoombie (1981) used finite element scheme with shift function. For nonlinear Klein-Gordon-equation, Kuo Pen-yu (1982) developed Galerkin method.

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(C) Spectral methods. Gazdag (1973), Tappert (1974) and Canosa & Gazdag (1977) employed finite Fourier method for K.D. V. equation. Shamel & Elsässer (1976), Watanabe, Ohishi & Tanaca (1977), Fornberg & Whitham (1978), Abe & Inoue (1980) and Kuo Pen-yu(1980) gave other schemes.

II Finite Difference Methods

We consider K. D. V. equation

$$\begin{cases} \dfrac{\partial U}{\partial t}+U\dfrac{\partial U}{\partial x}+\dfrac{\partial^3 U}{\partial x^3}=0, & x\in\mathbb{R},\ t>0, \\ U(x,0)=U_0(x), & x\in\mathbb{R}. \end{cases} \tag{1}$$

For simplicity we assume $U(x+1,t)=U(x,t)$, in this paper.

Let $I=[0,1]$, $Nh=1$, N is positive integer. $I_h=\text{set}\{x|x=jh,\ 1\leqslant j\leqslant N\}$.

Let $U^h(x,t)$ be approximate solution of $U(x,t)$. $U^h_z(x,t)$, $U^h_{\bar z}(x,t)$ and $U^h_{\hat z}(x,t)$ are respectively forward, backward and central difference quotient with respect to z. $z=x$ or t.

The first problem for constructing finite difference schemes is to approximate nonlinear terms suitably in order to simulate the properties of solution of (1) and avoid nonlinear instability of computation. Scott, Chu & Mclughlin (1973) showed the relation between the stability of motion and conservations. Lax (1976) pointed out that the solution of (1) has infinite conservations, such as

$$\int_0^1 U(x,t)dx=\int_0^1 U(x,0)dx, \tag{2}$$

$$\int_0^1 U^2(x,t)dx=\int_0^1 U^2(x,0)dx, \tag{3}$$

etc.

Similarly a reasonable finite difference scheme usually possesses discrete conservation (see courant, Friedrichs & Lewy (1928), Kuo Pen-yu (1965), Richtmyer & Morton (1967) and Morton (1977)). Since it is very difficult to construct a scheme whose solution keeps all conservations, so it is better to choose some of them.

If we want to simulate (2), then we have the following scheme

$$\begin{cases} \dfrac{\partial U^h}{\partial t}(x,t)+F^h_{\hat x}(x,t)+U^h_{x\bar x\hat x}(x,t)=0, & x\in I_h, t>0, \\ U^h(x,0)=U^h_0(x), & x\in I_h, \end{cases} \tag{4}$$

where $F^h(x,t)=\frac{1}{2}[U^h(x,t)]^2$.

Clearly

$$h\sum_{x\in I_h}U^h(x,t)=h\sum_{x\in I_h}U^h_0(x), \tag{5}$$

for practical computation, we must make the scheme (4) fully discrete. Let τ be mesh size of variable t, $t_k=k\tau$, k is non-negative integer, then we get

$$U_t^h(x,t_k)+\sigma F_{\hat{x}}^h(x,t_{k+1})+(1-\sigma)F_{\hat{x}}^h(x,t_k)+\sigma U_{x\bar{x}\hat{x}}^h(x,t_{k+1})+(1-\sigma)U_{x\bar{x}\hat{x}}^h(t_k)=0,$$
$$x\in I_h,\quad k\geqslant 0, \tag{6}$$

where $0\leqslant\sigma\leqslant 1$. If $\sigma=0$, then (6) is explicit scheme. If $\sigma\neq 0$, then (6) is implicit. For both stability and saving work, Greig & Morris (1976) employed hopscotch technique, i. e.

$$\sigma=\begin{cases}0, & \text{if } j+k \text{ is odd},\\ 1, & \text{if } j+k \text{ is even}.\end{cases}$$

Vliegenthart (1977) proposed the following dissipative scheme

$$U_t^h(x,t_k)+F_{\hat{x}}^h(x,t_k)+U_{x\bar{x}\hat{x}}^h(x,t_k)-\frac{h^2}{2\tau}U_{x\bar{x}}^h(x,t_k)=0,\quad x\in I_h, k\geqslant 0, \tag{7}$$

Vliegenthart considered two-order accurate dissipative scheme too. If we prefer to simulate both (2) and (3), we can use the following scheme

$$\begin{cases}\dfrac{\partial U^h}{\partial t}(x,t)+J^h(U^h,U^h)(x,t)+U_{x\bar{x}\hat{x}}^h(x,t)=0, & x\in I_h, t>0,\\ U^h(x,0)=U_0^h(x), & x\in I_h,\end{cases} \tag{8}$$

where

$$J^h(V^h,W^h)=\frac{1}{3}W^hV_{\hat{x}}^h+\frac{1}{3}(W^hV^h)_{\hat{x}}.$$

Then as pointed out by Zabusky & Kruskal (1965), the solution of (8) satisfies both (5) and

$$\|U^h(t)\|^2=\|U_0^h\|^2,$$

where

$$\|U^h(t)\|^2=h\sum_{x\in I_h}[U^h(x,t)]^2.$$

For practical computation, we use the following scheme

$$\begin{cases}U_t^h(x,t_k)+J^h(U^h+\sigma\tau U_t^h,U^h)(x,t_k)+(U^h+\sigma\tau U_t^h)_{x\bar{x}\hat{x}}(x,t_k)=0, & x\in I_h, k\geqslant 0,\\ U^h(x,0)=U_0^h(x), & x\in I_h,\end{cases} \tag{9}$$

then we obtain

$$\|U^h(t_{k+1})\|^2=\|U^h(t_k)\|^2+\tau(1-2\sigma)\|U_t^h(t_k)\|^2.$$

Clearly $\sigma\geqslant\frac{1}{2}$ implies $\|U^h(t_k)\|\leqslant\|U_0^h\|$, which means that the nonlinear instability is avoided. Particularly, if $\sigma=\frac{1}{2}$, then $\|U^h(t_k)\|=\|U_0^h\|$.

It should be noted that the energy conservation avoids only the instability in L_2-norm usually, since the L^∞-norm may be unbounded as $h\to 0$, (see Griffiths, Mitchell & Morris (1982)).

For improvement of accuracy, we usually adopt multiple-level schemes, such as

$$\begin{cases}U_{\hat{t}}^h(x,t_k)+J^h(U^h,U^h)(x,t_k)+U_{x\bar{x}\hat{x}}^h(x,t_k)=0, & x\in I_h, k\geqslant 1,\\ U_t^h(x,0)+J^h(U^h,U^h)(x,0)+U_{x\bar{x}\hat{x}}^h(x,0)=0, & x\in I_h,\\ U^h(x,0)=U_0^h(x), & x\in I_h,\end{cases} \tag{10}$$

Indeed, (10) identifies the scheme by Zabusky and Kruskal (1965). Another important technique to obtain stable and high order accurate scheme is to use prediction-correction scheme by Kuo Pen-yu and Wu Hua-mo (1981), such as

$$\begin{cases} \hat{U}^h(x,t_k) = U^h(x,t_k) - \beta\tau J^h(U^h,U^h)(x,t_k) - \beta\tau U^h_{x\bar{x}\hat{x}}(x,t_k), & x\in I_h, k\geqslant 0, \\ U^h_t(x,t_k) + J^h(U^h+\sigma\tau U^h_t, \hat{U}^h)(x,t_k) + (U^h+\sigma\tau U^h_t)_{x\bar{x}\hat{x}}(x,t_k) = 0, & x\in I_h, k\geqslant 0, \\ U^h(x,0) = U^h_0(x), & x\in I_h. \end{cases} \tag{11}$$

The technique used in (8) can be applied to other equations with soliton and soliton-like solutions. For instance we consider the following equation

$$\frac{\partial U}{\partial t} + aU^p\frac{\partial U}{\partial x} - \gamma\frac{\partial^2 U}{\partial x^2} + \beta\frac{\partial^{2r+1}U}{\partial x^{2r+1}} + mU|U|^q = 0, \tag{12}$$

where p, q, r are non-negative integers, a, β, γ and m are constants. If $r=p=q=0$, γ and β are real numbers, a and m are image numbers, then (12) becomes Hirota equation. (See Hirota, Suzuki (1973)). If $q=2, a=\beta=0, \gamma$ and m are image numbers, then (12) turns to nonlinear Schrödinger equation. If $\gamma=m=0$, a and β are real numbers, then (12) is generalized K. D. V. equation. If a,β and m are real numbers, $\gamma>0$, then (12) is generalized K. D. V-Burgers equation.

One of schemes for solving (12) is the following

$$\frac{\partial U^h}{\partial t}(x,t) + a\hat{J}^h(U^h,U^h)(x,t) - \gamma U^h_{x\hat{x}}(x,t) + \beta U^h_{\underbrace{x x\cdots x}_{r}\underbrace{\bar{x}\cdots\bar{x}}_{r}\hat{x}}(x,t) + mU^h|U^h|^q(x,t) = 0, \tag{13}$$

where

$$\hat{J}^h(U^h,W^h) = \frac{1}{p+1}(W^h)^pV^h_{\hat{x}} + \frac{1}{p+1}((W^h)^pV^h)_{\hat{x}}.$$

Another equation is R. L. W. equation

$$\begin{cases} \frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} + U\frac{\partial U}{\partial x} - \beta\frac{\partial^3 U}{\partial x^2\partial t} = 0, & x\in I,\ t>0, \\ U(x,0) = U_0(x). & x\in I, \end{cases} \tag{14}$$

where $a\geqslant 0, \beta>0$.

The eariest scheme is proposed by Peregrine (1966)

$$U^h_t(x,t_k) + \frac{1}{2}(a+U^h(x,t_k))(U^h_{\hat{x}}(x,t_k) + U^h_{\hat{x}}(x,t_{k+1})) - \beta U^h_{x\bar{x}t}(x,t_k) = 0.$$

Eilbeck & McGuire (1977) constructed another scheme later.

The second problem is the stability conditions of difference schemes and the error estimations. There are two main ways. The first one is Fourier method (See Richtmyer & Morton (1967)). Let $\tilde{U}^h(x,t)$ and $\tilde{U}^h_0(x)$ be the error of $U^h(x,t)$ and $U^h_0(x)$ respectively, $\tilde{f}(x,t)$ be the error of right term in difference scheme. Then from (6), we get the error equation approximately

$$\tilde{U}^h_t(x,t_k) + \sigma U^h(u,t_{k+1})\tilde{U}^h_{\hat{x}}(x,t_{k+1}) + (1-\sigma)U^h(x,t_k)\tilde{U}^h_{\hat{x}}(x,t_k)$$
$$+ \sigma\tilde{U}^h_{x\bar{x}\hat{x}}(x,t_{k+1}) + (1-\sigma)\tilde{U}^h_{x\bar{x}\hat{x}}(x,t_k) = \tilde{f}(x,t_k).$$

We take $U^h(x,t_k)$ and $U^h(x,t_{k+1})$ as constants, then the above equation is linear. If it is stable for all x, t and $U(x,t)$ considered, then we say that the scheme (6) is approximately stable. By using Fourier analysis, we get the stability conditions in table 1, where $M=\max_{x,t}|U(x,t)|$.

scheme	Greig & Morris	Vliegenthart	Zabysky & Kruskal
stability conditions	$\tau\leqslant\frac{h^3}{\lvert 2-h^2M\rvert}$	$\tau\leqslant\frac{h^3}{4+h^2M}$	$\tau\leqslant\frac{h^3}{4+h^2M}$

Table 1

Kuo Pen-yu (1965) proposed a technique for strict error estimation of nonlinear differenece scheme. This technique has been used for scheme (8). Let

$$\tilde{\rho}(t)=\|\tilde{U}_0^h\|^2+\int_0^t\|\tilde{\gamma}^h(\xi)\|^2d\xi.$$

Kuo Pen-yu & Sanz-Serna (1981) proved the following result.

Theorem 1 There exist positive constants N_1 and N_2 such that for all $t\geqslant 0$.

$$\|\tilde{U}^h(t)\|^2\leqslant N_1e^{N_2t}\tilde{\rho}(t),$$

where N_1 and N_2 depend only on $U^h(x,t)$.

Let $\tilde{\gamma}^h(x,t)$ be the truncation error of scheme (8), then there exist positive constants M_1 and M_2 such that

$$\|U(t)-U^h(t)\|^2\leqslant M_1e^{M_2t}\tilde{R}(t),$$

where M_1 and M_2 depend only on $U(x,t)$,

$$\tilde{R}(t)=\|U_0-U_0^h\|^2+\int_0^t\|\tilde{\gamma}^h(\xi)\|^2d\xi.$$

Therefore the scheme (8) is convergent provided

$$\tilde{R}(t)\to 0,\qquad \text{as } h\to 0.$$

The similar theoretical results have been obtained for schemes (11), (13) and the difference scheme for solving R.L.W. equation given by Wu Hua-mo & Kuo Pen-yu (1983).

The second equation is nonlinear Klein-Gordon equation

$$\begin{cases}\dfrac{\partial^2U}{\partial t^2}-\dfrac{\partial^2U}{\partial x^2}-U+U^3=0, & x\in I,\ t>0,\\ \dfrac{\partial U}{\partial t}(x,0)=U_1(x),\quad U(x,0)=U_0(x) & x\in I.\end{cases}\tag{15}$$

Let

$$E(t)=\int_0^1\left\{\left(\frac{\partial U}{\partial t}(x,t)\right)^2+\left(\frac{\partial U}{\partial x}(x,t)\right)^2-U^2(x,t)+\frac{1}{2}|U(x,t)|^4\right\}dx,$$

then

$$E(t)=E(0).\tag{16}$$

In order to simulate (16), Strauss & Vazquez (1978) defined the difference operator A^h:

$$A^hU^h(x,\ t_k)=\frac{1}{4}[(U^h(x,t_{k+1}))^2+(U^h(x,t_{k-1}))^2][U^h(x,t_{k+1})+U^h(x,t_{k-1})],$$

and proposed the following scheme

$$\begin{cases} U^h_{t\bar{t}}(x,\ t_k)-U^h_{x\bar{x}}(x,\ t_k)-\frac{1}{2}(U^h(x,t_{k+1})+U^h(x,t_{k-1}))+A^hU^h(x,t_k)=0, x\in I_h,\ k\geqslant 1, \\ U^h_t(x,0)=U^h_1(x),\ U^h(x,0)=U^h_0(x), \qquad x\in I_h. \end{cases} \tag{17}$$

Let

$$E^h(t_k)=-\frac{1}{2}\|U^h(t_k)\|^2-\frac{1}{2}\|U^h(t_{k+1})\|^2+\frac{1}{4}\|U(t_k)\|^4_{l_4}+\frac{1}{4}\|U(t_{k+1})\|^4_{l_4}$$
$$+\frac{1}{2}|U^h(t_k)|^2_1+\frac{1}{2}|U^h(t_{k+1})|^2_1+\|U^h_t(t_k)\|^2-\frac{\tau^2}{2}|U^h_t(t_k)|^2_1,$$

where

$$\|U(t_k)\|^4_{l_4}=h\sum_{x\in I_h}|U^h(x,t_k)|^4,\quad |U(t_k)|^2_1=h\sum_{x\in I_h}[U^h_x(x,t_k)]^2,$$

then

$$E^h(t)=E^h(0).$$

Another important equation with soliton is Sine-Gordon equation

$$\frac{\partial^2U}{\partial t^2}-\frac{\partial^2U}{\partial x^2}=-\sin U. \tag{18}$$

Perring & Skyrme (1962) gave the first scheme based on characteristic.

$$\begin{cases} U^h(x,t_{k+1})=U^h(x,t_k)+\tau\dot{U}^h(x,t_{k+\frac{1}{2}}), \\ \dot{U}^h(x,t_{k+\frac{1}{2}})=\dot{U}^h(x,t_{k-\frac{1}{2}})+\tau(U^h_{x\bar{x}}(x,t_k)-\sin U^h(x,t)), \end{cases} \tag{19}$$

where

$$t_{k+\frac{1}{2}}=\left(k+\frac{1}{2}\right)\tau.$$

Now let

$$E_1(t)=\int_0^1\left[\left(\frac{\partial U(x,t)}{\partial t}\right)^2+\left(\frac{\partial U}{\partial x}(x,t)\right)^2-2\cos U(x,t)\right]dx,$$

then

$$E_1(t)=E_1(0).$$

We can generalize the technique of Strauss and Vazquez to give the following scheme

$$\begin{cases} U^h_{t\bar{t}}(x,\ t_k)-U^h_{x\bar{x}}(x,t_k)=\dfrac{\cos U^h(x,t_{k+1})-\cos U^h(x,t_{k-1})}{U^h(x,t_{k+1})-U^h(x,t_{k-1})}, & x\in I_h,\ k\geqslant 1, \\ U^h_t(x,0)=U^h_1(x),\ U^h(x,0)=U^h_0(x), & x\in I_h, \end{cases} \tag{20}$$

then

$$E^h_1(t_k)=E^h_1(0),$$

where

$$E_1^h(t_k)=\frac{1}{2}|U^h(t_k)|_1^2+\frac{1}{2}|U^h(t_{k+1})|_1^2+\|U_t^h(t_k)\|^2$$

$$-\frac{\tau^2}{2}|U_t^h(t_k)|_1^2-h\sum_{x\in I_h}[\cos U^h(x,t_{k+1})+\cos U^h(x,t_k)].$$

The strict error estimation was obtained.

Ablowitz, Kruskal & Ladik (1979) considered a more general equation

$$\frac{\partial^2 U}{\partial t^2}-\frac{\partial^2 U}{\partial x^2}+F'(U)=0.$$

They put $W(x)=\frac{\partial U}{\partial t}(x,0)$ and $\tau=h$, then used the following scheme

$$V^h(x,0)=\frac{1}{2}(U_0(x)+U_0(x+h))+\frac{h}{4}(W(x)+W(x+h))$$

$$-\frac{h^2}{8}F'\left(\frac{U_0^h(x)+U_0^h(x+h)}{2}\right),\qquad x\in I_h$$

$$U^h(x,t_{k+1})=-U^h(x,t_k)+V^h(x,t_k)+V^h(x-h,t_k)$$

$$-\frac{h^2}{4}F'\left(\frac{V^h(x,t_k)+V^h(x-h,t_k)}{2}\right),\qquad x\in I_h,\ k\geqslant 0,$$

$$V^h(x,t_{k+1})=-V^h(x,t_k)+U^h(x+h,t_{k+1})+U^h(x,t_{k+1})$$

$$-\frac{h^2}{4}F'\left(\frac{U^h(x+h,t_{k+1})+U^h(x,t_{k+1})}{2}\right),\qquad x\in I_h,\ k\geqslant 0.$$

Many numerical experiments showed that the finite difference schemes for wave equations have the following advantages:

(i) Calculation is simple, especially for explicit scheme.

(ii) We can construct various schemes based on the properties of original partial differential equation such as conservations and characteristic. The solutions of such schemes usually possess the corresponding discrete properties.

(iii) The numerical results are quite accurate usually.

(To be contirued.)