第5卷 第3期
1985年7月

数学研究与评论
JOURNAL OF MATHEMATICAL
RESEARCH AND EXPOSITION

Vol.5 No.3
July, 1985

Some New Results on Degree Theory*

Chen Wenyuan (陈文嫄) and *Fan Xianling* (范先令)

(Lanzhou University)

An important contribution in the development of nonlinear analysis was given by Leray and Schauder. In paper [21] they extended the Brouwer degree to compact continuous fields (i. e. compact continuous perturbations of the identity) and consequently provided a very powerful tool for the investigation of various problems of nonlinear differential equations and nonlinear integral equations. The detailed report of the Leray-Schauder degree and its applications can be found in many books, including [20, 23, 7], to mention only a few.

Let Ω be an open subset of Banach space X, $f:\bar{\Omega}\to X$ a compact continuous field and $p\in X\setminus(\partial\Omega)$. In general terms, a Leray-Schauder degree $\deg(f, \Omega, p)$ is an integer with the following properties:

(D_1) If $p\in\Omega$, then the degree of identity *id* equal to 1.

(D_2) (Additive) If Ω_1 anb Ω_2 are disjoint open subsets of Ω with $p\notin f(\bar{\Omega}\setminus(\Omega_1\cup\Omega_2))$, then $\deg(f, \Omega, p)=\deg(f, \Omega_1, p)+\deg(f, \Omega_2, p)$.

(D_3) (Homotopy invariance) If $H:[0,1]\times\bar{\Omega}\to X$ is a compact continuous mapping, θ: $[0,1]\to X$ is a continuous mapping and $\theta(t)\notin h_t(\partial\Omega)$ where $h_t(x)=x-H(t,x)$, then $\deg(h_t,\Omega,\theta(t))$ is constant independent of $t\in[0,1]$.

By means of degree theory, a lot of consequence which are very important in theory as well as in application can be derived, such as, a series of fixed point theorems, multiplicity theorem, alternative theorem, bifurcation theorem and domain invariant theorem etc.

Unfortunately there are many problems in which the mapping can't be reputes as a compact continuous field. Thus, it is natural to attempt to extend the basic concepts of degree theory to more broad context. During the past few years many authors, perhaps the earlist is Browder [3, 4], are devoted to this subject. The purpose of this paper is to sketch those recent development and their connection

* Received Jan. 14, 1983.

(*) Research supported under CSASF Grant Math-386.

with the earlier result. Especially we shall list some unpublished researches to this subject made by Fan and others in Lanzhou University during the last two years.

1. There seems little doubt that the concept multi-valued mapping will prove important in the future development of our subject. For example, the study of mathematical economic and control theory leads to consider the fixed points of multi-valued mappings. The degree theory for upper semicontinuous compact fields with compact and convex values is due to Cellina-Lasota [6]. A systematic investigation of general multi-valued compact fields and their connection with algebraic topology is due to Borisovich-Gel'man-Muschkis-Obuhovskii[2]. References for a full discussion have been given in the bibliographic notes of above literature and thus not be reproduced here.

2. Elworthy and Tromba [11] have dereloped degree theory for smooth Fredholm mappings of index zero between Banach manifolds. They proved that this degree has the properties (D_1) - (D_3) above except that (D_3) hold in a weaken form, namely, under a suitably restricted class of deformation H, the degree deg (f,Ω,p) is not necessary invariant but $|\deg(f,\Omega,p)|$ is. These results have been redone in a somewhat simple way under more weaken restrict by Borisovich-Zvyagin-Sapronov [1]. Such results promise to have applications in singular integral equations and general nonlinear elliptic differential equations. The Fredholm map of positive index also have been treated in [12]. The corresponding degree is no longer an integer but a Pontrjagin framed cobordism class.

3. The set-contract mapping and condensing mapping based on the idea of noncompactness can be regarded as a natural extension of compact continuous mapping. First, Nassbaum [28, 29] established the degree for k$<$1 set-contract fields and condensing fields. Analogous results for cone preserving and its applications can be found in Yu [36, 37, 38]. Petryshyn [29] contains various applications of degree of 1-set-contract fields.

Further discussions are as follows. The degree theory for single-valued and multi-valued limit compact fields is developed in Sadovskii[32] and webb[34] respectively. An attempt to extend the degree theory to multi-valued ultimately compact fields can be found in Petryshyn-Fitzpatick[31] and in Duc-Thanh-Ang [10], It is worth while point out that these degrees deg$(f,\Omega,\ p)$ may not uniquely be determined by their values on $\partial\Omega$, namely, the property of boundary value dependence is not true for limit compact and ultimately compact fields, as the example in[28] shows.

4. The class of A-proper mappings arising from the method of finitedimensional approximation is a wide class of nonlinear mappings. It contains for instance compact continuous fields, ball-condensing fields, mappings of types (s), (s_+) and

(α) as its subclass. Browder and Petryshyn [5, 30] discussed systematically its properties and definite its degree with respect to a given approximation scheme and Wong [35]refine this degree with the aid of the filters. General speaking, this degree isn't an integer, it is a set of integer together with $+\infty$ and $-\infty$, so we call it generalized degree, e. g. if $0\in\Omega$, we have $\deg(id, \Omega, 0)=\{1\}$ and $\deg(-id, \Omega, 0)=\{+1, -1\}$ with respect to the ordinary approximation scheme. This generalized degree was extended to multi-valued A-proper in 1976 by Massobò-Nistri. But in their paper [24]the theorem of homotopy invariance is wrong. Fan[13]corrected this mistake and yielded a theorem of homotopy invariance in weaken form(see below 7). Therefore, a great number of assertions in Petryshyn [30] are transplanted by Fan [14] to multi-valued case. (Notice, difference from the single-valued case, an upper semicontinuous multi-valued A-proper mapping restrict to a bounded closed set need not be proper).

Skrupnik [33] and Fan [16] discussed the degree for bounded demicontinuous mappings of type (α) which are the special case of A-proper mappings. It is proven that this degree is an integer, independs on choice of injective approximation scheme and prossesses uniqueness. Furthermor, in this case, the Hopf theorem is valid.

5. Fitzpatrick [19] defined generalized degree for uniform limits of single-valued A-proper mappings, including compact mappings, nonexpansive fields, 1-ball contract fields, pseudo-and quasi-monotone mappings as special example. According to the similar idea, Feng and Li [18, 22] yielded the degree for monotone mappings, Min in his master thesis [25] yielded the degree for pseudo-monotone mappings, Fan[15]discussed the degree for uniform limits (in more general sense) of multi-valued A-proper mapping.

6. Finally, the theory of homotopy extension of mapping between difference spaces X and Y was studied by Qin, Yu and Chen [8, 9]. More recently Fan [17] made a unified treatment of A-proper mapping and ultimately compact mapping via the extensive theorem, he esablished the degree $\deg(f, \Omega, p)$ for multi-valued mapping $f=A+F$: $\bar{\Omega}\subset X\to 2^Y$, here A is A-proper and F is A-ultimately-compact. (a term himself coined);

7. (Appendix) The theorem of homotopy invariance property in Massabò-Nistri [24] as follows:

Let H: $\bar{\Omega}\times[0,1]\to 2^Y$ be a multi-valued map such that $H_t=H(\cdot,t)$ is a family of A-proper maps which depend continuously on t, ($H_{t_n}\to H_t$ in the sense of $d^*(G_{H_{t_n}}, G_{H_t})=\sup\{\mathrm{dis}(z, G_{H_t})\,|\,z\in G_{H_{t_n}}\}\to 0$ when $t_n\to t$ where $G_{H_{t_n}}$ and G_{H_t} are the graphs of H_{t_n} and H_t respectively). If $p\notin H(\partial\Omega\times[0,1])$ then $\deg(H_t, \Omega, p)$ is independent of t in $[0,1]$.

The following example shows that this theorem is not true.

Example. Let $X=Y=l^2$, $\Omega=\{x\in X \mid \|x\|<1\}$.

The approximation scheme is taken as usual orthonal projection scheme. $H:\bar{\Omega}\times[0,1]\to 2^Y$ is defined as follows:

when $t\neq 0$ put $H(x,t)=-x$ for any $x\in\bar{\Omega}$.;

when $t=0$ put

$$H(x,0)=\begin{cases} x. & \text{either } x=(\xi_1,\xi_2,\cdots)\in\Omega \text{ such that the set}\{\xi_i \mid \xi_i\neq 0\}\text{is finite or } \|x\|=1, \\ \bar{\Omega}. & x=(\xi_1,\xi_2,\cdots)\in\Omega \text{ such that the set } \{\xi_i \mid \xi_i\neq 0\}\text{is infinite.} \end{cases}$$

It is easy to verify that H satisfies all the conditions in the above-mentioned theorem but $\{1\}=\deg(H_0,\ \Omega,\ 0)\neq\deg(H_1,\ \Omega,\ 0)=\{\pm 1\}$.

The following theorem is proved by Fan [13].

Theorem Let X, Y be Banach spaces with approximation scheme$(X_n,\ Y_n;\ P_n,\ Q_n)$, Ω an open bounded subset of X and $H:\bar{\Omega}\times[0,\ 1]\to 2^Y$ a multi-valued mapping with nonempty convex values. If H satisfies the following conditions:

i) $(H_t)_n:\bar{\Omega}_n\to 2^{Y_n}$ is upper semicontinuous with compact values for each positive integer n and each $t\in[0,1]$where$(H_t)_n(x)=Q_nH(p_nx,t)$ and $\Omega_n=\Omega\cap X_n$;

ii) for each $n\in N$ $d^*(G_{(H_t)_n},\ G_{(H_{t_0})_n})\to 0$ as $t\to t_0$;

iii) for any seguence $\{n_j\}\subset N$ with $n_j\to\infty$ as $j\to\infty$ and corresponding sequences $\{x_{n_j}\}$, $\{t_{n_j}\}$ and $\{y_{n_j}\}$ such that $x_{n_j}\in\partial\Omega_{n_j}$, $t_{n_j}\in[0,\ 1]$, $y_{n_j}\in(H_{t_{n_j}})_{n_j}(x_{n_j})$and an element $y_0\in Y$ such that $\|y_{n_j}-Q_{n_j}y_0\|\to 0$ as $n_j\to\infty$ there exist subsequences$\{x_{n_{j(k)}}\}\subset\{x_{n_j}\}$ and$\{t_{n_{j(k)}}\}\subset\{t_{n_j}\}$such that $P_{n_{j(k)}}\ x_{n_{j(k)}}\to x_0, t_{n_{j(k)}}\to t_0$ and $y_0\in H(x_0,\ t_0)$.

iv) $p\in Y\setminus H(\partial\Omega\times[0,1])$

then $\deg(H_t,\ \Omega,\ 0)$is independent of t in $[0,\ 1]$.

References

[1] Borisovich, Yu. G., Zvyagin, V. G., Sapronov, Yu. I., Nonlinear Fredlolm maps and the Leray-Schauder theorem, *Usp. Mat. Nauk.* 32(1977)3—54.(Russian).

[2] Borisovich, Yu. G., Gel'man, B. D., Muschkis, A. D., Obuhovskii, V. V., Topological methods in the fixed point theory of multivalued mappings, *Usp. Mat. Nauk.* 35 (1980)59—126.(Russian).

[3] Browder, F. E., Nassbaum, R. D., The topological degree for noncompact nonlinear mappings in Banach spaces, *Bull. Amer. Math. Soc.* 74(1968)4, 671—676.

[4] Brower, F. E., Petryshyn, V. V., The topological degree and Galerkin approximations for noncompact operators on Banach spaces, *Bull. Amer. Math. Soc.* 74(1968)4. 641—646.

[5] Browder, F. E., Petryshyn. W. V., Approximation method and the generlized topological degree for nonlinear mappings in Banach spaces, *J.Funct. Anal.* 3(1969)217—245.

[6] Cellina, A., Lasota, A., A new approach to the definition of topological degree for multivalued mappings, *Atti. Accad. Naz. Lincei Rend. CL. Sci. Fis. Mat. Natus.* 47(1969) 8.434-440.

[7] Chen, W. Y., Nonlinear Functional Analysis (1982) Gansu Remin(Chinese).

[8] Chen, W. Y., Qin, C. L., Compact perturbations of a continuous mapping, *J. Math. Res. Expos.*(1981)Initial Issue, 39—46, (Chinese).

[9] Chen, W. Y., Yu, O. Y., A note about essentiality and inessentiality of compact fields, Nonlinar Analysis, TMA. (to appear).

[10] Duc, D. M., Thanh, D. N., Ang, D. D., Relative topological degrees of set-valued compact vector fields and its applications, *J. Math. Anal. Appl.* 80(1981). 406—432.

[11] Elworthy, K., Tromba, A., Differential structures and Fredholm maps, *Proc. Symp. Pure Math., Global Anal*, vol 15(1970)45-94.

[12] Elworthy, K., Tromba, A., Degree theory on Banach manifolds, *Proc. Symp. Pure Math. Nonlinear Functional Analysis*, vol 18(1970) 86—94.

[13] Fan, X.L., A note on generalized degree for generalized gradient mappings, (to appear).

[14] Fan, X.L., On generalized degree theory for multivalued A-proper mappings (to appear).

[15] Fan, X.L., On generalized degree theory for uniform limits of multivalued A-proper mappings, (to appear).

[16] Fan, X.L., A topologieal degree for multivalued mappings of type (α), (to appear).

[17] Fan, X.L., A generalized degree for multivalued A-ultimately compact vector fields, (to appear).

[18] Feng, D.X., Li, S.J., Topological degree of monotone mappings, *Acta Math. Sinica*, 24 (1981)1, 106-115. (Chinese).

[19] Fitzpatrick, P.M., A generalized degree for uniform limits of A-proper mappings, *J. Math. Anal. Appl.* 35(1971) 536-552.

[20] Krasnosel'skii, M.A., Zabreiko, P.P., Geomertrical methods in nonlinear analysis, Nauka, Moskow(1975). (Russian).

[21] Leray, J.,Schauder, J., Topologie et equations fonctionelles, *Ann. Sci, Ecol. Norm. Sup.* 51(1934), 45—78.

[22] Li, S. J., Feng, D. X., Topological degree of the multivalued maximal monotone operator in Hilbert spaces, *Acta Math. Sinica*, 25(1982)5, 533—542. (Chinese).

[23] Lloyd, N. G., Degree theory, Cambridge tracts in math. No. 73(1978) Cambridge Univ. Press, London.

[24] Massabò, I., Nistvi, P., A topological degree for multivalued A-proper maps in Banach spaces, *Boll. U. M. I.* 13—B(1976)5, 672—685.

[25] Min, L. Q., A topological degree for quasi-bounded psuedo-monotone mappings(to appear).

[26] Nassbaum, R, D., The fixed point index and fixed point theorems for k-set-contractions, Doctoral dissertation, University of Chicago, Chicago(1969).

[27] Nassbaum, R. D., Degree theory for local condensing maps, *J. Math. Anal. Appl.* 37(1972)3 741—766.

[28] Pascale, E., Guzzardi, R., On the boundary values dependence for the topological degree of multivalued noncompact maps, *Boll. U. M .I.* 3—A(1976)1, 110—116.

[29] Petryshyn, W. V., Fixed point theorems for various classes of l-set-contractive and l-ball-contractive mappings in Banach spaces, *Tran. Amer. Math. Soc.* 182(1973)323—352.

[30] Petryshyn, W. V., On the approximation-solvability of equations involving A-proper and psuedo-A-proper mappings, *Bull. Amer. Math. Soc.* 80(1975)223—312.

[31] Petryshyn, W. V., Fitzpatrick, P. M., A degree theory, fixed point theorems, and mapping theorems for multivalued noncompact mappings. *Tran. Ames. Math. Soc.* 194(1974)1—25.

[32] Sadovskii, B. N., Limit compact and condensing mappings, *Usp. Mat. Nank.*,27(1972)81—146, (Russian).

[33] Skrupnik, I. V.. Nonlinear elliptic equations of higher order, Naukova Dumka, kiev(1973) (Russian).

[34] Webb, J. R. L., On degree theory for multivalued mappings and applications, *Boll. U.M.I.* 9(1674), 137—158.

[35] Wong, H. S. F., The topological degree of A-proper maps, *Canad. J. Math.*, 23 (1971)3, 401—412.

[36] Yu, Q. Y., Condensing maps in ordered Banach spaces and their positive fixed points, *J. Lanzhou University*(Natur. sci. ed.)2(1979)23—32(Chinese).

[37] Yu, Q. Y., Monotonic condensing maps and their fixed points, *Acta Math. Sinica*,24(1981)3, 430—435(Chinese).

[38] Yu, Q. Y., On the non-zero solution of the equation $x-T(x)=\lambda F(x)$ (to appear).