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Concerning Various Combinatorial Identities*

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1. Introduction

According to H. W. Gould [3] the notation S: p/q indicates a kind of binomial coefficient summation in which p denote the number of binomial coefficients appearing in the numerator term of the summation, and q has the same meaning for the denominator term. In order to classify combinatorial identities (binomial coefficient summations) more precisely, we may introduce the concept "degree" for a combinatorial identity of the type S: p/q, say. We may say that an identity for the sum of form S: p/q has degree r if there are r independent parameters involved in the formula. Accordingly we may denote its type by the notation S: $[p/q;\ r]$. Thus, instance, Nanjundiah's identity

$$\sum_{k=0}^{n}\binom{m-x+y}{k}\binom{n+x-y}{n-k}\binom{x+k}{m+n}=\binom{x}{m}\binom{y}{n} \tag{1. 1}$$

is a formula of type S: $[3/0;\ 4]$ since there are four parameters x, y, m and n involved, where x and y may be any real or complex numbers inasmuch as the both sides of (1. 1) are just polynomials in x and y with same degrees.

The object of this paper is to exhibit a number of combinatorial identities for sums of the forms S: $3/0$; $4/0$; $p/0$; $0/p$ and p/p, respectively. In particular, we shall show that various identities contained in the Table 6 Gonld's formulary [3] are actually equivalent to (1. 1), or implied by (1. 1) as particular cases.

Generally, combinatorial identities of the same type S: $[p/q;\ r]$ may sometime be equivalent to each other irrespective of how distinct in their appearances. To be precise, let us state a simple and useful criterion as follows.

Criterion Suppose that A and B are two combinatorial identities of the same type S: $[p/q;\ r]$. Then A and B are equivalent formulas if the set of r free parameters of A (or B) can be obtained from of B (or A) by linear substitutions. More-

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over, if A is of type S: $[p/q; s]$ with $s<r$ and if the set of s free parameters of A may be expressed linearly in terms of that of B, then A is a genuine special case of B.

Applying this criterion to (1. 1) we easily find that the following two combinatorial identities are equivalent to (1. 1):

$$\sum_{k=0}^{n}(-1)^k\binom{x+y+1}{k}\binom{x+n-k}{m-k}\binom{y+m-k}{n-k}=\binom{x}{m}\binom{y}{n} \tag{1. 2}$$

and

$$\sum_{k=0}^{t}\binom{t}{k}\binom{k+x}{t+s}\binom{s}{r-t+k}=\binom{x}{r}\binom{x-r+t}{s-r+t}. \tag{1. 3}$$

If fact, (1. 2) and (1. 3) are both of type S: [3/0; 4] and can be obtained from (1, 1) by substitutions $x=-m'-1,\ y=y',\ m=x'-m',\ n=n'$ and $x=x',\ m=r,\ y=x'-r+t,\ n=s-r+t$, respectively.

Notice that (1, 2) is a symmetrical identity from which (6. 48) (Zeitlin-Surànyi's identity), (6. 16) and (6. 31) of Gould's [3] may be derived by the aid of the following substitutions

$x=x',\ y=y',\ m=n=n';$

$x=n'+j,\ y=-j-1,\ m=r+n',\ n=x'-j;$

$x=y=-x'+n'-1,\ m=n=n',$

respectively. Moreover, (6. 52) (Stanley's identity) and (6. 37) of [3] are deducible from (1. 1) with the substitutions

$x=x'+a,\ y=y'+b,\ m=x'+a-b,\ n=y'+b-a;$

and

$x=y=3n',\ m=n=n'$

respectively. Since (6. 52) is of type S: [3/0; 4], it clear that Stanley's identity and (1. 1) are equivalent.

2. Identities for Sums of Forms S: 3/0 and S: 4/0

As may easily be verified, the identities due to Surânyi, Bizley, Riordan, Krall, Gould and Andrews [1] (*i. e*, the formulas (6. 19), (6. 42), (6. 43), (6. 45), (6. 47), and (6. 51), as listed in Gould's formulary (loc. cit.)) are all deducible from (1. 3) by suitable choice of the parameters x, r, s, and t. Actually (6. 19), (6.30), (6.32), (6.42), (6.43), (6.45), (6.51) of [3] may be obtained by taking

(i)	$x=x'+r',\quad t=n,\quad s=r',\quad r=r',$	(Surânyi)
(ii)	$x=x',\quad r=s=t=n,$	
(iii)	$x=x'+n,\quad r=s=t=n,$	
(iv)	$x=a,\quad r=b-d,\quad s=c,\quad t=b,$	(Bizley)
(v)	$x=a,\quad r=b+c-d,\quad s=c,\quad t=b,$	(Bizley)
(vi)	$x=x',\quad r=n,\quad s=m,\quad t=n,$	(Riordan)
(vii)	$x=x'+n+r',\quad r=n,\quad s=r',\quad t=n,$	(Gould)

respectively.

As regards Krall's identity and Anderws' extension of Riordan's identity of the forms

$$\sum_{k=0}^{b}\binom{b}{k}\binom{r+a+b-k}{n-k}\binom{r+a-n}{r-k}=\binom{a+r}{r}\binom{a+b}{n} \tag{2.1}$$

and

$$\sum_{k=0}^{n}\binom{m-\mu}{k}\binom{n+\mu}{k+\mu}\binom{\nu+\mu+k}{m+n}=\binom{\nu+\mu}{m}\binom{\nu}{n} \tag{2.2}$$

respectively, one may derive then from (1. 3) by taking

(iix) $x=a+r,\quad r=r,\quad s=a+r-n,\quad t=b,$ (Krall)

(ix) $x=\nu+\mu,\quad r=m,\quad s=n+\mu,\quad t=m-\mu,$ (Andrews)

respectively.

Obviously, Bizley's, Krall's, and Andrews' identities are all of the type S:[3/0, 4]. Thus in accordance with our criterion we may assert that the following proposition is true.

Proposition Nanjundiah's, Stanley's, Bizley's, Krall's and Andrews' identities of the type S: [3/0; 4] are equivalent to each other. In other words, they are practically the same identity. On the other hand, Surànyi's, Riordan's and Gould's identities are genuine special cases of (1. 1) or of (1. 2).

For the q-binomial coefficients defined by

$$\begin{bmatrix} x \\ n \end{bmatrix}=\prod_{j=1}^{n}\frac{q^{x-j+1}-1}{q^{j}-1},\qquad \begin{bmatrix} x \\ 0 \end{bmatrix}=1,$$

it is known that Gould has established a q-analog of Nanjundiah's formula (1. 1), viz.

$$\sum_{k=0}^{n}\begin{bmatrix} m-x+y \\ k \end{bmatrix}\begin{bmatrix} n+x-y \\ n-k \end{bmatrix}\begin{bmatrix} x+k \\ m+n \end{bmatrix}q^{(n-k)(m-x+y-k)}=\begin{bmatrix} x \\ m \end{bmatrix}\begin{bmatrix} y \\ n \end{bmatrix}. \tag{2.3}$$

Similarly we can obtain a symmetrical q-analoy of (1. 2) by starting from Gould's formula (2. 3) with suitable substitutions of variables and transformations, namely, we have

$$\sum_{k=0}^{n}(-1)^{k}\begin{bmatrix} x+y+1 \\ k \end{bmatrix}\begin{bmatrix} x+n-k \\ m-k \end{bmatrix}\begin{bmatrix} y+m-k \\ n-k \end{bmatrix}q^{\frac{1}{2}k(k-1)-mn}=\begin{bmatrix} x \\ m \end{bmatrix}\begin{bmatrix} y \\ n \end{bmatrix}. \tag{2.4}$$

Of course (2. 3) and (2. 4) are also equivalent.

Here it may be of some interest to note that an algebraic identity due to E. W. Wright [5] can easily be converted into a formula of type S: [4/0; 4], namely

$$\sum_{k=c}^{d}\binom{r+s+t-u-k}{r+s-u}\binom{r+s-u}{r+k}\binom{r-u+k}{k-u}\binom{s}{k}=\binom{r+s-u}{s}\binom{s+t-u}{t}\binom{t+r-u}{r}, \tag{2.5}$$

where $c=\max\{0,\ u\}$, $d=\min\{r,\ s,\ t\}$ and $r,\ s,\ t,$ and u are non-negative integers.

As a final remark it may be worth mentioning that our criterion can be used systematically to discover more equivalent relations among existing combinatorial identities. But this may require much more time to do.

3. Identities for Sums of Forms S: p/0 and S: 0/p

Let x and y be any real numbers and let $\theta_1,\cdots,\theta_p$ denote the distinct roots of the equation $z^p-1=0$. Then using mathematical induction on n one can verify the following pair of combinatorial identities

$$\sum_{k=0}^{n}(-1)^{kp}\binom{\theta_1 x}{k}\cdots\binom{\theta_p x}{k}=\binom{n-\theta_1 x}{n}\cdots\binom{n-\theta_p x}{n}. \tag{3.1}$$

$$\sum_{k=0}^{n}\frac{1}{\binom{k+1-\theta_1 x}{k}\cdots\binom{k+1-\theta_p x}{k}}=\frac{x^p-1}{x^p}\left\{1-\frac{1}{\binom{n+1-\theta_1 x}{n+1}\cdots\binom{n+1-\theta_p x}{n+1}}\right\}. \tag{3.2}$$

These are formulas of types S: [$p/0$; 2] and S: [$0/p$,2] respectively. We shall omit their inductive proofs here since the induction involves only pure algebraic calculations.

As may readily be observed, identities (1. 4), (1. 5), (1. 25) and (3. 16) of Gould's formulary [3] are particular cases of (3. 1); and (2. 1), (2. 2), (2. 11) and etc. are consequences of (3. 2). Of particular interest we mention the following special instances

$$\sum_{k=0}^{n}\binom{x}{k}\binom{-x}{k}=\binom{n+x}{n}\binom{n-x}{n}. \tag{3.3}$$

$$\sum_{k=0}^{\infty}\binom{x}{k}\binom{-x}{k}=\frac{\sin(\pi x)}{\pi x},\qquad x\neq 0. \tag{3.4}$$

$$\sum_{k=0}^{\infty}\binom{x}{k}\binom{-x}{k}\binom{x\sqrt{-1}}{k}\binom{-x\sqrt{-1}}{k}=\frac{1}{2}\left(e^{\pi x}-e^{-\pi x}\right)\frac{\sin(\pi x)}{\pi x}. \tag{3.5}$$

$$\sum_{k=0}^{n}\frac{1}{\binom{k+1+x}{k}\binom{k+1-x}{k}}=\frac{x^2-1}{x^2}\left\{1-\frac{1}{\binom{n+1+x}{n+1}\binom{n+1-x}{n+1}}\right\},\quad (x\neq 0). \tag{3.6}$$

The infinite series summations (3. 4) and (3. 5) are obtained by letting $n\to\infty$ in the corresponding formulas of the types S: [2/0; 2] and S: [4/0; 2] respectively.

4. A Formula of Type S: [p/p; 3] and Its Consequences

What we want to present here is the following identity

$$\sum_{k=0}^{n}\frac{\binom{k-\theta_1 x}{k}\cdots\binom{k-\theta_p x}{k}}{\binom{k+1-\theta_1 y}{k}\cdots\binom{k+1-\theta_p y}{k}}=\frac{1-y^p}{x^p-y^p}\left\{1-\frac{\binom{n+1-\theta_1 x}{n+1}\cdots\binom{n+1-\theta_p x}{n+1}}{\binom{n+1-\theta_1 y}{n+1}\cdots\binom{n+1-\theta_p y}{n+1}}\right\}, \tag{4.1}$$

where $|x|\neq|y|$ and $\theta_1,\cdots,\theta_p$ are the p-th roots of unity. This is a formula of type S: [p/p; 3] and can also be proved by induction on n. Obviously formula (4. 1) of Gould's [3] is the simple special case of the above (4. 1) with $p=1$.

For $p=2$ we have the following identities

(4. 2)
$$\sum_{k=0}^{n} \frac{\binom{k+x}{k}\binom{k-x}{k}}{\binom{k+1+y}{k}\binom{k+1-y}{k}} = \frac{1-y^2}{x^2-y^2}\left\{1-\frac{\binom{n+1+x}{n+1}\binom{n+1-x}{n+1}}{\binom{n+1+y}{n+1}\binom{n+1-y}{n+1}}\right\},$$

(4. 3)
$$\sum_{k=0}^{\infty} \frac{\binom{k+x}{k}\binom{k-x}{k}}{\binom{k+1+y}{k}\binom{k+1-y}{k}} = \frac{1-y^2}{x^2-y^2}\left\{1-\frac{y\sin(\pi x)}{x\sin(\pi y)}\right\},$$

where formula (4. 3) is obtained from (4. 2) by letting $n\to\infty$. These are of types S: [2/2; 3] and S: [2/2; 2] respectively. In particular (4. 3) implies (with $x=0$) the following formula of type S: [0/2; 1]

(4. 4)
$$\sum_{k=0}^{\infty} \frac{1}{\binom{k+1+y}{k}\binom{k+1-y}{k}} = \frac{y^2-1}{y^2}\left\{1-\frac{\pi y}{\sin(\pi y)}\right\}.$$

Moreover, letting $y\to 0$ in (4. 4) we get Euler's equality

$$\sum_{k=0}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

5. A Pair of Reciprocal Formulas for Double Sums

Reciprocal formulas are often useful for obtaining combinatorial relations. Here we offer the following reciprocal pair as an extension of the well-known self-reciprocity of the δ-transform to double sums:

(5. 1)
$$g(n, k) = \sum_{0\leqslant i+j\leqslant n} \binom{n}{i, j}(-1)^i f(i, k+sj)$$

and

(5. 2)
$$f(n, k) = \sum_{0\leqslant i+j\leqslant n} \binom{n}{i, j}(-1)^i g(i, k+sj),$$

where each double summation is taken over all non-negative integers i and j such that $i+j\leqslant n$, and s is any positive integer, and $\binom{n}{i, j}$ is defined by

$$\binom{n}{i, j} = \binom{n}{i}\binom{n-i}{j} = \frac{n!}{i!\, j!\,(n-i-j)!}.$$

The reciprocity between (5. 1) and (5. 2) means that the set of equations (5. 1) with $n=0, 1, 2, \cdots$ can be deduced from that of (5. 2) with $n=0, 1, 2, \cdots$ and vice versa. Since (5. 1) and (5. 2) are symmetrical in form it suffices to verify (5. 1)$\Rightarrow$(5. 2). Now let (5. 1) be true for all n and k. Then after substitution of (5. 1) into the right-hand side of (5. 2) we can simplify the expressions by the aid of Vandermonde's convolution formula and the orthogonality relation for binomial coefficients, thus finally arriving at the result $f(n, k)$ which is precisely the left-hand side of (5. 2).

The particular case of (5. 1)$\Leftrightarrow$(5. 2) with $s=1$ may be used to obtain various interesting relations among Fibonacci numbers $F_n(n=0,1,2,\cdots)$ or among other special sequences of numbers. For instance, making use of the relation for Fibonacci numbers

$$F_k+F_{k+1}+F_{k+18}=47F_{k+10},$$

and applying the reciprocal formulas (5. 1) and (5. 2) to

$$f(i,\ j)=(-1)^iF_{i+18j},$$

we find

$$F_{10n+18k}=\sum_{0\leqslant i+j\leqslant n}\binom{n}{i,\ j}47^{-n}F_{i+18(j+k)},$$

$$F_{n+18k}=\sum_{0\leqslant i+j\leqslant n}\binom{n}{i,\ j}(-1)^{n+i}47^iF_{10i+18(j+k)}.$$

Similarly, we have

$$F_{10n+k}=\sum_{0\leqslant i+j\leqslant n}\binom{n}{i,\ j}47^{-n}F_{18i+j+k},$$

$$F_{18n+k}=\sum_{0\leqslant i+j\leqslant n}\binom{n}{i,\ j}(-1)^{n+i}47^iF_{10i+j+k},$$

if we take $f(i,\ j)=(-1)^iF_{18i+j}$. Some details about these and the like have been illustrated in a previous paper [4].

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