第5卷 第4期
1985年10月

数学研究与评论
JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION

Vol. 5 No. 4
Oct. 1985

On the Generalized Upwind Scheme for the Neutron Transport Equation*

Huang Mingyou (黄明游)

(Jilin University)

Let Ω be a convex domain in the (x, y) plane with boundary Γ. Denote by $n=(n_x, n_y)$ the outward unit normal to Γ. Consider the following problem

$$\mu \frac{\partial u}{\partial x}+\nu\frac{\partial u}{\partial y}+\sigma u=f \quad \text{in } \Omega \tag{1}$$

$$u=g \quad \text{on } \Gamma_-=\{(x, y)\in\Gamma;\ \mu n_x+\nu n_y<0\}, \tag{2}$$

where μ, ν are real parameters and $\sigma=\sigma(x, y)$ satisfies $\sigma\geqslant\sigma_0>0$. Equation (1) arises in neutron transport theory. In this paper, we study theoretically the stability and convergence properties of a discontinuous Galerkin approximation to problem (1)、(2), which is interprated as a generalized upwind scheme with arbitrary meshes for the first order hyperbolic equation. By means of the energe method, an error estimate in L_2 norm is proved which is a half order higher in mesh size h than that given in [1].

Let $\{\mathcal{T}_h, 0<h<1\}$ be a family of triangulations. $\{\mathcal{T}_h\}$ consists of triangular elements denoted by K. We assume that $\{\mathcal{T}_h\}$ is quasiuniform in the sense that there exists a constant $\alpha>1$ such that

$$h(K)\leqslant\alpha\rho(K) \quad \text{for all } K\in\mathcal{T}_h$$

where $h(K)$ = diameter of K, $h=\max\limits_{K\in\mathcal{T}_h}\{h(K)\}$, $\rho(K)$ = sup{diameter of all the circles contained in K}. Let $K\in\mathcal{T}_h$, ∂K be the boundary of K. We define

$$\partial K_-=\{(x, y)\in\partial K;\ \beta\cdot n(x, y)<0\},\quad \partial K_+=\partial K\backslash\partial K_-,$$

where $\beta=(\mu, \nu)$, $n=(n_x, n_y)$ is the outward unit normal to ∂K. And the following notations are employed

$$(w, v)_K=\int_K w\cdot v\,dxdy,\quad \langle w, v\rangle_{\partial K}=\int_{\partial K} w\cdot v n\beta\,ds$$

$$\|v\|_K^2=\int_K v^2\,dxdy,\quad |v|^2_{\partial K}=\int_{\partial K} v^2|n\beta|\,ds.$$

*Received Dec. 18, 1982.

We use the finite dimensional space

$$S_h=\{v\in L_2(\Omega);\ v|_K\in P_r(K),\ \forall K\in\mathscr{T}_h\}$$

where $P_r(K)$ is the set of polynomials of degree$\leqslant r$ on K, then the discontinuous Galerkin approximation to problem (1)、(2) is defined as follow: Find $u_h\in S_h$ such that

$$(3)\qquad \begin{cases}\sum\limits_{K\in\mathscr{T}_h}\{-(u_h,\ v_\beta)_K+\langle\tilde{u}_h,\ v\rangle_{\partial K}\}+(\sigma u_h,\ v)=(f,\ v),\ \forall v\in S_h\\ \tilde{u}_h=g \quad \text{on } \Gamma_-\end{cases}$$

where $(\cdot,\ \cdot)$ denotes the inner product in $L_2(\Omega)$, $v_\beta=\mu\dfrac{\partial v}{\partial x}+\nu\dfrac{\partial v}{\partial y}$, and $\tilde{u}_h$ is the upwind value of u_h on ∂K defind by

$$\tilde{u}_h=\begin{cases}u_h^+ & \text{the interior trace of } u_h \text{ on } \partial K_+,\\ u_h^- & \text{the exterior trace of } u_h \text{ on } \partial K_-.\end{cases}$$

In the simplist case $r=0$, i. e. S_h consists of piecewise constants, the method (3) is similar to the ordinary upwind difference scheme. Therefore we call method (3) in general $(r\geqslant 0)$ as the generalized upwind scheme. This method defines a series of explicit schemes.

Introduce bilinear form

$$B(w,\ v)=\sum_{K\in\mathscr{T}_h}\{-(w,\ v_\beta)_K+\langle w,\ v\rangle_{\partial K}\}+(\sigma w,\ v),$$

then (3) can be written as

$$(4)\qquad B(u_h,\ v)=(f,\ v),\ \forall v\in S_h.$$

Let v be a piecewise function on $\mathscr{T}_h$ with convention $v^-=0$ on Γ_+. A usefull identity, namely

$$(5)\qquad B(v,\ v)=\frac{1}{2}\sum_s\int_s[v]^2|n\beta|\,ds+(\sigma v,\ v)-\frac{1}{2}\int_{\Gamma_-}(v^-)^2|n\beta|\,ds$$

can be verified, where the summation $\sum\limits_s$ is taken over all sides of the triangulation $\mathscr{T}_h$ and $[v]=v^+-v^-$.

It is easy to prove by applying (5) the following stability result.

Theorem 1 The discrete problem (3) has a unique solution u_h for any given $f\in L_2(\Omega)$ and $g\in L_2(\Gamma_-)$, and u_h satisfies

$$\sum_s\int_s[u_h]^2|n\cdot\beta|\,ds+\|u_h\|^2\leqslant C(\|f\|^2+|g|^2_{\Gamma_-}),$$

where constant C does not depend on h.

From Theorem 1 we know that the solutions $\{u_h\}$ of (3) is uniformly bounded with respect to h in L_2. So there exists a subsequence $\{u_{h_i}\}$ which is weakly convergent to a function u^* in L_2. The following theorem tells us that this limit function u^* is a weak solution of (1)、(2) defined by:

$$-(u,\ v_\beta)+(\sigma u,\ v)=(f,\ v)-\int_{\Gamma_-} g\cdot vn\beta ds$$

for any $v\in H^1(\Omega)$ such that $v|_{\Gamma_-}=0$.

Theorem 2 Assume that $\{u_h\}$ is weakly convergent to u^* in $L_2(\Omega)$ when $h\to 0$. Then u^* must be a weak solution of (1)、(2). Consequently, the problem (1)、(2) does exist a weak solution.

Now turn to the erroe estimate of u_h. To do this, we introduce a mesh dependent norm

$$|||v|||=\left\{\|v\|^2+\sum_{K\in\mathscr{T}_h} h\|v_\beta\|_K^2+\sum_s\int_s [v]^2|n\beta|ds\right\}^{\frac{1}{2}},$$

and the following improved stability estimate is needed.

Lemma Assume that v is any a piecewise continuous function on $\mathscr{T}_h$ with values $v^-|_\Gamma=0$. Then there exist constants κ_0 and C independent of h and v such that for $0<\kappa<\kappa_0$ and h small enough

$$\kappa|||v|||^2\leqslant C\{B(v,v+\kappa hv_\beta)+\kappa^2h^2\sum_{K\in\mathscr{T}_h}\int_{\partial K}|v_\beta|^2|n\beta|ds\}.$$

In terms of this lemma we are able to prove the following error estimate.

Theorem 3 Assume that u and u_h are the solutions of (1)、(2) and (3) respectively, and that $u\in H^{r+1}(\Omega)$, $r\geqslant 0$. Then for small h

$$|||u-u_h|||\leqslant Ch^{r+\frac{1}{2}}\|u\|_{H^{r+1}(\Omega)}.$$

The proofs of all results mentioned in this short paper are given in a author's research report [2]. And an application of the upwind type finite element scheme to some nonlinear transport equation has studied in [3].

References

[1] Lesaint, P., and Raviart, P. -A., On a finite element method for solving the neutron transport equation, Mathematical Aspects of the F. E. M. in Patial Differential Equations, Ed. by C. de Boor, Acad. Press, New York, 1974.

[2] Huang, M. Y., On the generalized upwind scheme for the neutron transport equation, *Research Report*, Jilin University, No. 8204, 1982.

[3] Huang, M. Y., A finite element approximation of upwind type to the vorticity transport equations in bounded domain, *Research Report*, Jilin University, No. 8303, 1983.