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"Markov chain must have a beginning"

In Memory of Prof. Loo-Keng Hua(华罗庚)*

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The proposition quoted in the title above is a folklore. It can be argued as follows. If a Markov chain has run an infinitely long time, then every "state" will have become a "steady state", and nothing will happen any more. This is physicists' talk, which sounds apt for the occasion. The formal definitions are given below.

Let $P=(p_{ij})$, $1\leqslant i, j\leqslant r$, be a finite stochastic (Markov) matrix; namely $p_{ij}\geqslant 0$ for all i, j; and $\sum_{j=1}^{r} p_{ij}=1$ for all i. Let V be the class of all r-valued probability distributions, written as row vectors:

$$v=(v_1, \cdots v_r); \qquad v_j\geqslant 0 \text{ for all } j, \quad \sum_{j=1}^{r} v_j=1.$$

It is trivial that if $v\in V$, then $vP^n\in V$ for all $n\geqslant 1$, where P^n denotes the nth power of P. Namely, P maps V into V. Let us say that v (in V) has *history* of length n iff there exists $u\in V$ such that $uP^n=v$. This terminology is justified because if v has history of length n then it has history of length k for all k, $1\leqslant k\leqslant n$, as is easily seen. Let us say that v is "periodic" (under P) of period d iff $vP^d=v$. when $d=1$. we say v is "invariant".

The late Professor Loo-Keng Hua (who was my teacher in Kunming, c. 1940) communicated to me in 1984, while he was at the California Institute of Technology, the following result.

Hua's proposition. A probability distribution cannot have an arbitrarily long history. unless it is periodic.

Hua formulated his result in terms of eigenvectors, and assumed implicitly that P has an inverse, so that time ranges over all integers. I showed his result to Zhongxin Zhao (赵忠信). who gave the above reformulation, and an apparently rather complicated proof. It turns out that a short proof can be given which allows for an easy extension. The interpretation given by Hua however becomes blurred in the ensuing proposition. I will also add an analogue for continuous time Markov chains, in which Hua's original claim (see below) will be retrieved.

Let W denote a class of uniformly bounded r-tuples; namely; there exists a number M such that for all v in W, we have

$$(1) \qquad \sup_{1\leqslant j\leqslant r} |v_j| \leqslant M.$$

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The extension mentioned above is from V to W. It is obvious that P maps W into W. The terms "history" and "periodic" defined above can be extended to W.

Proposition 1. There exists a positive integer d determined by P, with the following property. If $v \in W$ and v has arbitrarily long history, then v is periodic with period d.

Proof. Consider the Markov chain with finite state space $(1, \cdots, r)$ and transition matrix P. Each state is either transient (nonrecurrent) or positive recurrent. The latter states fall into classes, and each class has a period (uniquely defined). This "period" is not to be confused with the period defined above. Let the least common multiple of all these periods by d. This is the period stated in the proposition. It is a consequence of the main limit theorem for Markov chains that we have, for all i and j:

(2) $$\lim_{n\to\infty} p_{ij}^{(nd)} = \pi_{ij} .$$

This result is contained in Section I.6 of [1]. But since the latter treats the more difficult case of a countably infinite state space and an infinite matrix P, the proof can be somewhat simplified. Rather surprisingly, so far as I can ascertain after consulting experts in the field, no explicit statement giving (2) can be found in the literature. It is a folklore. [Mathematicians who try to obtain a result such as (2) without a preliminary classification of the states would be baffled. Samuel Eilenberg asked me a similar question some years ago.] Now let $v^{(n)} \in W$ and

(3) $$v^{(n)} P^{nd} = v .$$

Such a $v^{(n)}$ exists because v has history of length nd, by hypothesis. By the compactness of W with respect to pointwise convergence, namely, the theorem of Bolzano-Weierstrass, there exists a sequence $n_i \to \infty$ such that

(4) $$v^{(n_i)} \to u \in W_\circ$$

Since $P^{n_i d} \to \Pi = (\pi_{ij})$ by (2), and we are dealing with finite sums, it follows that

(5) $$v^{(n_i)} P^{n_i d} \to u\Pi ,$$

and consequently by (3)

(6) $$u\Pi = v.$$

Now it follows from (2) that

(7) $$\Pi P^d = \Pi.$$

Combining (6) and (7) we obtain

(8) $$v = u\Pi = u\Pi P^d = vP^d.$$

In Hua's letter to me he actually claimed that d = 1, so that v is invariant. The preceding proof shows that this is the case if every (positive) recurrent state has period 1. [Is this the only case?] Now if the Markov chain runs in continuous time, so that $p(t) = (p_{ij}(t))$, $t \geqslant 0$, is its standard transition semigroup, then we have in place of (2), for all i and j:

$$(9) \qquad \lim_{t\to\infty} p_{ij}(t) = \pi_{ij} .$$

This result can be extracted from Section II. 10 of [1] with a little effort. There is no "period" in continuous time. A proof completely similar to the above then yields the following.

Proposition 2. Suppos $v \in W$, and for a sequence of values of t going to infinity we have $v(t)$ in W such that $v(t)P(t) = v$. Then we have for all $t \geqslant 0$:

$$vP(t) = v.$$

Reference:

[1] K. L. Chung, Markov chains with stationary transition probabilities, Grundlehren der mathematischen Wissenschaften, Band 104, Second edition, Springer-Verlag, 1967.

开莱我兄教授：

来美之后还未作书问安，歉何如之。

现有一事请教：以下的事实是否有人证明过。

"Markov链必有开始。除去以下一个特例。若 $p=(p_1,\dots,p_n)$ 是 Markov 方阵的不变向量，即

$$pA=p$$

因此

$$\dots,\ pA^{-1},\ p,\ pA,\ p^2A,\ \dots,\ p^nA$$

一致。如果 p 不是 A 的特征向量，必有一 ℓ_0 使 $pA^{-\ell_0}$ 出现负分量。

如果用这事实，一个数学模型如果不能解释过去，如何能希望它来解决未来。（当 A 是很 primitive 的）。但 ℓ_0 代表什么！如果还未有人做过，当将证明及其应用寄上。敬请

教安　　华罗庚

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为了纪念华罗庚教授，本刊特发表1984年华老访美期间给钟开莱教授关于马氏链的通信和钟先生的纪念文章。