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The Characterizations of Tensor Product Functors and Their Derived Functors of Left Modules

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In 1979, Zhou Boxun[1] introduced concepts of the tensor product of left modules, so that there exist tensor products of left modules which is an extension of the tensor products of modules over commutative rings. In [2], it was proved that $-\otimes M$ preserves direct limits and $(-\otimes M, \mathrm{Hom}(M, -))$ is an adjoint pair. In this paper, we characterize functor $-\otimes M$ and its derived functors. These results can be consider as extensions of Watts' theorems[4].

Let K be a commutative ring, R and S be K-rings. All the rings and the modules are unitary in this paper.

Theorem 1. Let $F: {}_R\mathfrak{M} \to {}_{R\otimes S}\mathfrak{M}$ be a covariant functor, then following statements are equivalent:

1) F preserves direct limits;

2) F is right exact and preserves sums;

3) $F \cong M\underset{R}{\otimes}-$ for some $M \in {}_{R\otimes S}\mathfrak{M}_R$, where $\underset{R}{\otimes}$ means the tensor product of R-modules in the ordinary sense;

4) $F \cong -\otimes\overline{M}$ for some $\overline{M} \in {}_S\mathfrak{M}$, where $\otimes$ means the tensor product of left $R\otimes S$-modules in the sense of [1];

5) There is functor $G: {}_{R\otimes S}\mathfrak{M} \to \mathfrak{M}_R$, such that (F, G) is an adjoint pair.

Proof.

1)⇒2). Cokenel and sum are direct limits.

2)⇒3). See "Remark" following theorem 3.33 in [4]. Here $M = FR$ can be constructed as right R-module and as a dimodule.

3)⇒4). Let $M = FR$, which is in ${}_{R\otimes S}\mathfrak{M}_R$ by 3). For any $L \in {}_R\mathfrak{M}$, R, S, L, M are all K-modules. Suppose that $\overline{M} = S\underset{K}{\otimes}M$, which is left S- and K-bimodule. From theorems 1.12, 1.13 in [4], the definition of the ring tensor

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product, associative law for $\otimes$ and proposition 1 in [2], we obtain following left $R\otimes S$-module isomorphisms:

$$
\begin{aligned}
{}_{R\otimes S}M_{R\underset{R}{\otimes}R}L &\cong ((R\otimes S)\otimes M)\underset{R}{\otimes}L \cong ((R\underset{K}{\otimes}S)\otimes M)\underset{R}{\otimes}L \cong ((R\underset{K}{\otimes}S)\underset{K}{\otimes}M)\underset{R}{\otimes}L \\
&\cong ((R\underset{K}{\otimes}(S\underset{K}{\otimes}M))\underset{R}{\otimes}L = (R\underset{K}{\otimes}\overline{M})\underset{R}{\otimes}L \cong (\overline{M}\underset{K}{\otimes}R)\underset{R}{\otimes}L \cong \overline{M}\underset{K}{\otimes}(R\underset{R}{\otimes}L) \cong \overline{M}\underset{K}{\otimes}L \\
&\cong L\underset{K}{\otimes}\overline{M} \cong L\otimes\overline{M}.
\end{aligned}
$$

In every step, isomorphism is natural for L. By 3) we have functor natural equivalence: $F\cong M\underset{R}{\otimes}- \cong -\otimes\overline{M}: {}_R\mathfrak{M}\to{}_{R\otimes S}\mathfrak{M}$.

4)⇒5). By theorem 2.2 in [3], thke $G=\mathrm{Hom}_S(M, -)$.

5)⇒1). By theorem 2.19 in [4]. q. e. d.

Theorem 2.

Let $F: {}_S\mathfrak{M}\to{}_{R\otimes S}\mathfrak{M}$ be a covariant functor, then following statments are equivalent:

1) F preserves direct limits;

2) F is right and preserves sums;

3) $F\cong L\underset{S}{\otimes}-$ for some $L\in{}_{R\otimes S}\mathfrak{M}_S$, where $\underset{S}{\otimes}$ means tensor product in ordinary sense.

4) $F\cong\overline{L}\otimes-$ for some $\overline{L}\in{}_R\mathfrak{M}$, where $\otimes$ means tensor product of left $R\otimes S$-module.

5) there is a functor $G: {}_{R\otimes S}\mathfrak{M}\to{}_S\mathfrak{M}$ such that (F, G) is an adjoint pair.

Proof.

2)⇒3). See "Remark" following theorem 3.33 in [4]. $L=FS$ can be constructed as a right S-module and as a bimodule.

3)⇒4). Let $L=FS$. From 3) implies that $L\in{}_{R\otimes S}\mathfrak{M}_S$. For any $M\in{}_S\mathfrak{M}$, suppose that $\overline{L}=R\underset{K}{\otimes}L$, so $\overline{L}$ is left R- and K-bimodule. According to the same reason as 3)⇒4) in theorem 1, we obtain also following left $R\otimes S$-module isomorphisms:

$$
\begin{aligned}
{}_{R\otimes S}L\underset{S}{\otimes}M &\cong ((R\otimes S)\otimes L)\underset{S}{\otimes}M \cong ((R\underset{K}{\otimes}S)\underset{K}{\otimes}L)\underset{S}{\otimes}M \cong ((S\underset{K}{\otimes}R)\underset{K}{\otimes}L)\underset{S}{\otimes}M \\
&\cong (S\underset{K}{\otimes}(R\underset{K}{\otimes}L))\underset{S}{\otimes}M = (S\underset{K}{\otimes}\overline{L})\underset{S}{\otimes}M \cong (\overline{L}\underset{K}{\otimes}S)\underset{S}{\otimes}M \cong \overline{L}\underset{K}{\otimes}(S\underset{S}{\otimes}M) \cong \overline{L}\underset{K}{\otimes}M \cong \overline{L}\otimes M.
\end{aligned}
$$

In every step, isomorphism is natural for M. By 3) we have functor natural equivence: $F\cong L\underset{S}{\otimes}- \cong \overline{L}\otimes-: {}_S\mathfrak{M}\to{}_{R\otimes S}\mathfrak{M}$.

4)⇒5). We can obtain a result similar theorem 2.1 in [3]:

For each $A\in{}_R\mathfrak{M}$, $C\in{}_S\mathfrak{M}$ and $M\in{}_{R\otimes S}\mathfrak{M}$, there is an abelian group isomorphism : $\mathrm{Hom}_{R\otimes S}(A\otimes C,\ M)\cong\mathrm{Hom}_S(C,\ \mathrm{Hom}_R(A,\ M))$, and for A, C, M are all naturally, where M is considered as left R-module, left S-module and bimodule. Now take $A={}_R L$ in 4), then $G: {}_{R\otimes S}\mathfrak{M}\to {}_S\mathfrak{M}$, $M\mapsto \mathrm{Hom}_R(L, M)$ and $F: {}_S\mathfrak{M}\to {}_{R\otimes S}\mathfrak{M}$, $C\mapsto L\otimes C$ construct as an adjoint pair (F, G).

The rest of this theorem is similar to theorem 1. q. e. d.

Now we give characterization of $\mathrm{Tor}_n(-, M)$ (see [3] Def. 5.1).

Theorem 3.

Let $\{T_n\}$ be a positive sequence of functors from ${}_R\mathfrak{M}$ to ${}_{R\otimes S}\mathfrak{M}$ ([4] p. 212), then $T_n\cong \mathrm{Tor}_n(-, M)$, $n=0, 1, \cdots$, iff

1) $\{T_n\}$ is strongly connected;
2) T_n preserves sums for all n;
3) T_0 is right exact;
4) $T_n R=O$ for all $n=1, 2, \cdots$.

Proof.

The necessity is in [3] §5. It only remains to prove the converse. By theprem 1, $T_0\cong -\otimes M\cong \mathrm{Tor}_0(-, M)$, where $M=T_0 R\in {}_{R\otimes S}\mathfrak{M}$. For all $L\in {}_R\mathfrak{M}$, there is a short exact sequence

$$O\longrightarrow H\longrightarrow F\longrightarrow L\longrightarrow O,$$

in which F is a free left R-module, because $\{T_n\}$ is strongly connected, so there is a long exact sequence

$$\cdots\to T_2L\to T_1H\to T_1F\to T_1L\to T_0H\to T_0F\to T_0L\to O.$$

By 2), 4), if $n\geqslant 1$, then $T_nF=0$. Further, since $\{\mathrm{Tor}_n(-, M)\}$ is strongly connected, there exists a long exact sequence

$$\cdots\to \mathrm{Tor}_2(L, M)\to \mathrm{Tor}_1(H, M)\to \mathrm{Tor}_1(F, M)\to$$
$$\mathrm{Tor}_1(L, M)\to H\otimes M\to F\otimes M\to L\otimes M\to 0.$$

Since $T_0\cong -\otimes M$, there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} O & \longrightarrow & T_1L & \longrightarrow & T_0H & \longrightarrow & T_0F \\ & & \downarrow{\scriptstyle f_L} & & \downarrow & & \downarrow \\ O & \longrightarrow & \mathrm{Tor}_1(L, M) & \longrightarrow & H\otimes M & \longrightarrow & F\otimes M \end{array}$$

The diagram-chasing shows f_L is a homomorphism, which is an isomophism by "5-Lemma". It can be verified that $f: T_1\to \mathrm{Tor}_1(-, M)$ is a naturally equivalent and f_L is independent of H and F. Eductive step shows $T_n\cong \mathrm{Tor}_n(-, M)$, $n=1, 2, \cdots$. q. e. d.

Similarly, we can characterize the sequence of $\{\mathrm{Tor}_n(L, -)\}$.

References

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