含有导数的黎曼型边值问题*

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本文讨论广义解析函数边界条件带有导数的黎曼型边值问题。对广义解析函数的黎曼问题有 Л. Г. Михайлов[1]、[2]闻国椿的工作[3]。著名数学家 И. Н. Векуа 在〔4〕中提出需要研究广义解析函数带有导数的黎曼型问题，并指出它在几何、力学方面的重要价值。

多连通区域 S^+ 是在围道 L_0 之内，包含原点，而在 L_1、$L_2\cdots L_m$ 之外，其相补区域为 $S^- = S_0^- + S_1^- + \cdots + S_m^-$，$S_0^-$ 为无界域。$L = L_0 + L_1 + \cdots + L_m$。并假定在映射 $\zeta = \frac{1}{z}$ 下，L_0 的影曲线 $\overset{*}{L}_0$ 仍为简单的封闭围道。

问题的提法：讨论下述一阶椭圆型复方程

$$\frac{\partial W}{\partial \bar{z}} + B(z)\overline{W} = 0, \quad z \in E \tag{1}$$

系数满足条件 $B(z) \in C_a^{n-1}(S^+ + L)$，$B(z) \in C_a^{p-1}(S_q^- + L_q)$，$q = 1, 2\cdots m$。$\frac{1}{\bar{z}^2} B(\frac{1}{z}) \in C_a^{p-1} \cdot (\overset{*}{S}_0^- + \overset{*}{L}_0)$。$L \in C_a^{N+1}$，$0 < a \leqslant 1$，$N = \max\{n, p\}$。

寻求方程（1）的分片正则解 $W(z) = \{W^+(z), W^-(z)\}$，其在L上满足边值条件：

$$\sum_{k=0}^{n}\left\{a_k(t)\frac{\partial^k W^+(t)}{\partial t^k} + \frac{1}{\pi i}\int_L \frac{R_k(t,\tau)}{\tau - t}\frac{\partial^k W^+(\tau)}{\partial t^k}d\tau\right\} - \sum_{k=0}^{p}\left\{b_k(t)\frac{\partial^k W^-(t)}{\partial t^k}\right.$$

$$\left. + \frac{1}{\pi i}\int_L \frac{S_k(t,\tau)}{\tau - t}\frac{\partial^k W^-(\tau)}{\partial \tau^k}d\tau\right\} = f(t), \quad t \in L, \tag{2}$$

$$\int_{L_0}\frac{W^-(\tau)}{\tau} = 0 \tag{3}$$

条件（3）相当于解析函数的黎曼问题要求在无穷远处为零的情况。$a_k(t)$、$b_k(t)$、$R_k(t,\tau)$、$S_k(t,\tau)$、$f(t)$ 均为L上H类函数，且满足条件：

$$b_p(t) - S_p(t,t) \neq 0, \quad t \in L, \quad a_n(t) + R_n(t,t) \neq 0, \quad t \in L。 \tag{4}$$

方程（1）在区域 S^+L 上的正则解则有下述表达式〔2〕、〔4〕

$$W^+(z) = \Phi^+(z) + \iint_{S^+}\Gamma_1(z, t, S^+)\phi^+(t)dT_t + \iint_{S^-}\Gamma_2(z, t, S^+)\overline{\phi^+(t)}dT_t, \quad z \in S^+ \tag{5}$$

即若 $W^+(z) \in C_a^n(S^+ + L)$，由于 $\Phi^+(z) = \frac{1}{2\pi i}\int \frac{W^+(\tau)d\tau}{\tau - z}$ 及 $L \in C_a^{N+1}$，故 $\Phi^+(z) \in C_a^n(S^+ + L)$。对区域 S_j^-，$j = 1, 2\cdots m$，亦有类似于（5）的预解表达式

$$W^-(z) = \Phi^-(z) + \iint_{S_j^-}\Gamma_1(z, t, S_j^-)\Phi^-(t)dT_t + \iint_{S_j^-}\Gamma_2(z, t, \overline{S_j^-)\Phi^-(t)}dT_t, \quad z \in S_j^- \tag{6}$$

对区域 S_0^- 上的 $W^-(z)$ 来说，令 $\zeta = \frac{1}{z}$，则方程（1）变为

$$\frac{\partial \overset{*}{W}}{\partial \bar{\zeta}} + \overset{*}{B}(\zeta)\overline{\overset{*}{W}} = 0, \quad \xi \in \overset{*}{S}_0^- \tag{7}$$

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此处 $\overset{*}{W}(\zeta)=W^{-}\left(\frac{1}{\zeta}\right)$，$\overset{*}{B}(\zeta)=-\frac{1}{\bar{\zeta}^{2}}B(\zeta)$，$\zeta\in\overset{*}{S}_0^{-}$。此时 $\overset{*}{S}_0^{-}+\overset{*}{L}$为有界单连区域对应方程（7）的预解式为

$$\overset{*}{W}(\zeta)=\overset{*}{\Phi}(\zeta)+\iint\limits_{\overset{*}{S}_0^{-}}\Gamma_1^{*}(\zeta,\eta,\overset{*}{S}_0^{-})\overset{*}{\Phi}(\eta)\,dT_\eta+\iint\limits_{\overset{*}{S}_0^{-}}\Gamma_2^{*}(\zeta,\eta,\overset{*}{S}_0^{-})\overline{\overset{*}{\Phi}(\eta)}\,dT_\eta,\quad \zeta\in\overset{*}{S}^{-}\qquad(8)$$

此处 $\overset{*}{\Phi}(\zeta)=\frac{1}{2\pi i}\int_{\overset{*}{L}_0}\frac{\overset{*}{W}(\tau)\,d\tau}{\tau-\zeta}$，$\zeta\in\overset{*}{S}_0$，故可得

$$W^{-}(z)=\Phi^{-}(z)+\iint\limits_{S_0^{-}}\Gamma_1(z,t,S_0^{-})\Phi^{-}(t)\,dT_t+\iint\limits_{S_0^{-}}\Gamma_2(z,t,S_0^{-})\overline{\Phi^{-}(t)}\,dT_t,\quad z\in S_0^{-}\qquad(9)$$

此处 $\Phi^{-}(z)=\overset{*}{\Phi}\left(\frac{1}{z}\right)=\frac{z}{2\pi i}\int_{L_0}\frac{W^{-}(\tau)\,d\tau}{\tau(\tau-z)}$，$\Phi^{-}(\infty)=\lim\limits_{z\to\infty}\Phi^{-}(z)=-\frac{1}{2\pi i}\int_{L_0}\frac{W^{-}(\tau)\,d\tau}{\tau}$，

$\Gamma_1(z,t,S_0^{-})=\Gamma_1^{*}\left(\frac{1}{z},\frac{1}{t},\overset{*}{S}_0^{-}\right)\frac{1}{|t|^4}$，$\Gamma_2(z,t,S_0^{-})=\Gamma_2^{*}\left(\frac{1}{z},\frac{1}{t},\overset{*}{S}_0^{-}\right)\frac{1}{|t|^4}$。

将无穷处为零的分片全纯函数 $\Phi(z)=\{\Phi^{+}(z),\Phi^{-}(z)\}$ 表示为

$$\Phi^{+}(z)=\frac{(-1)^n}{(n-1)!}\sum_{j=1}^{m}\frac{1}{2\pi i}\int_{L_j}\frac{u(\tau)}{(\tau-z_j)^n}(\tau-z)^{n-1}\ln\left(1-\frac{\tau-z_j}{z-z_j}\right)d\tau+$$

$$+\frac{(-1)^n}{(n-1)!}\frac{1}{2\pi i}\int_{L_0}u(\tau)(\tau-z)^{n-1}\ln\left(1-\frac{z}{\tau}\right)d\tau+P_{n-1}(z),\quad z\in S^{+}\qquad(10)$$

此处 z_j 为S_j^{-}中某一固定点，$P_{n-1}(z)$ 为 $n-1$ 次多项式。

$$\Phi^{-}(z)=\frac{(-1)^p}{(P-1)!}\frac{1}{2\pi i}\sum_{j=1}^{m}\int_{L_j}u(\tau)\frac{\prod_{k=1}^{m}{}'(\tau-z_k)^n}{\tau^{p+mn}}(\tau-z)^{p-1}\ln\left(1-\frac{\tau-z_j}{z-z_j}\right)d\tau+$$

$$+\frac{(-1)^p}{(P-1)!}\frac{1}{2\pi i}\int_{L_0}u(\tau)\frac{\prod_{k=1}^{m}(\tau-z_k)^n}{\tau^{p+mn}}(\tau-z)^{p-1}\ln\left(1-\frac{z}{\tau}\right)d\tau+\sum_{k=0}^{p-1}A_k z^k\ln z+$$

$$+\sum_{k=1}^{mn}\frac{B_k}{z^k}+\sum_{k=0}^{p-1}d_{qk}z^k,\quad z\in S_q^{-},\quad q=1,2\cdots m\qquad(11)$$

$\prod_{k=1}^{m}{}'$ 表示乘积没有 $k=j$ 的因子。

$$\Phi^{-}(z)=\frac{(-1)^p}{(P-1)!}\frac{1}{2\pi i}\int_{L_0}u(\tau)\frac{\prod_{k=1}^{m}(\tau-z_k)^n}{\tau^{p+mn}}(\tau-z)^{p-1}\ln\left(1-\frac{\tau}{z}\right)d\tau+\sum_{k=1}^{mn}\frac{B_k}{z^k}+$$

$$+\sum_{k=0}^{p-1}d_{0k}z^h,\quad z\in S_0^{-}\qquad(12)$$

对应于方程（1）的预解核具有下述性质：

$$\left.\begin{aligned}&\Gamma_1(z,t,S)=\frac{1}{\pi}\overline{B(t)}\Omega_2(z,t,S),\quad \Gamma_2(z,t,S)=\frac{1}{\pi}B(t)\Omega_1(z,t,S)\\&\Omega_1(z,t,S)=\frac{1}{t-z}+O(\ln|t-z|),\quad \Omega_2(z,t,S)=O(\ln|t-z|)\end{aligned}\right\}\qquad(13)$$

此处区域S可以是 S^{+}，S_q^{-}（$q=1,2\cdots$）。

将（10）代入（5）交换积分次序得

$$W^{+}(z)=\frac{(-1)^{n}}{(n-1)!}\sum_{j=1}^{m}\frac{1}{2\pi i}\int_{L_j}\frac{u(\tau)}{(\tau-z_j)^{n}}(\tau-z)^{n-1}\ln\left(1-\frac{\tau-z_j}{z-z_j}\right)d\tau+$$

$$+\frac{(-1)^{n}}{(n-1)!}\frac{1}{2\pi i}\int_{L_0}u(\tau)(\tau-z)^{n-1}\ln\left(1-\frac{z}{\tau}\right)d\tau+$$

$$+\frac{1}{2\pi i}\int_{L}H_1(z,\tau)u(\tau)d\tau+\frac{1}{2\pi i}\int_{L}H_2(z,\tau)\overline{u(\tau)}d\tau+\sum_{k=0}^{2n-1}C_kW_k^{+}(z),\ z\in S^{+} \tag{14}$$

此处

$$H_1(z,\tau)=\begin{cases}\dfrac{(-1)^{n}}{(n-1)!}\displaystyle\iint_{S^-}\Gamma_1(z,t,S^-)(\tau-t)^{n-1}\ln\left(1-\frac{t}{\tau}\right)dT_t, & \tau\in L_0\\[2ex] \dfrac{(-1)^{n}}{(n-1)!}\displaystyle\iint_{S^-}\Gamma_1(z,t,S^+)\frac{(\tau-t)^{n-1}}{(\tau-z_j)^{n}}\ln\left(1-\frac{\tau-z_j}{\tau-z_j}\right)dT_t, & \tau\in L_j,\ j=1,2\cdots m\end{cases}$$

$$H_2(z,\tau)=\begin{cases}\dfrac{(-1)^{n-1}}{(n-1)!}\displaystyle\iint_{S^-}\Gamma_2(z,t,S^+)\overline{(\tau-t)^{n-1}\ln\left(1-\frac{t}{\tau}\right)}dT_t, & \tau\in L_0\\[2ex] \dfrac{(-1)^{n-1}}{(n-1)!}\displaystyle\iint_{S^-}\Gamma_2(z,t,S^+)\overline{\frac{(\tau-t)^{n-1}}{(\tau-z_j)^{n}}\ln\left(1-\frac{\tau-z_j}{t-z_j}\right)}dT_t, & \tau\in L_j,\ j=1,2\cdots m\end{cases}$$

$$k!\,W_{2k}^{+}(z)=z^{k}+\iint_{S^-}\Gamma_1(z,t,S^+)t^{k}dT_t+\iint_{S^-}\Gamma_2(z,t,S^+)\overline{t^{k}}dT_t\ ,\quad z\in S^{+}$$

$$k!\,W_{2k+1}^{+}(z)=iz^{k}+\iint_{S^-}\Gamma_1(z,t,S^+)it^{k}dT_t-\iint_{S^-}\Gamma_2(z,t,S^+)it^{k}dT_t,\quad z\in S^{+}$$

对于（14）中的第一项，其 1 到 $n-2$ 阶导数的核函数是连续的，$n-1$ 阶导数出现对数型奇异性，n 阶导数出现柯西核。下面证明 $H_2(z,\tau)$ 其 1 到 $n-1$ 阶导数核函数是连续的，n 阶导数出现对数型奇异性。对 $H_1(z,\tau)$ 来说，其 1 到 n 阶导数对应的核均是连续的。注意到（13），$\tau\in L_0$ 时有

$$H_2(z,\tau)=\frac{(-1)^{n-1}}{(n-1)!}\frac{1}{\pi}\iint_{S^-}\frac{B(t)}{t-z}\overline{(\tau-t)^{n-1}\ln\left(1-\frac{t}{\tau}\right)}dT_t+\cdots\cdots \tag{15}$$

讨论核的奇异性的阶，只要讨论前面部分。为了简便记

$$B_1(t,\tau)=B(t)\overline{(\tau-t)^{n-1}\ln\left(1-\frac{t}{\tau}\right)}\ ,\quad K_1(z,\tau)=\frac{1}{\pi}\iint_{S^-}\frac{B_1(t,\tau)}{t-z}dT_t\ ,$$

$$\frac{\partial K_1}{\partial z}=\frac{1}{\pi}\iint_{S^-}\frac{B_1(t,\tau)dT_t}{(t-z)^2}=-\frac{1}{\pi}\iint_{S^-}\frac{\partial}{\partial t}\left[\frac{B_1(t,\tau)}{t-z}\right]dT_t+\frac{1}{\pi}\iint_{S^-}\frac{1}{t-z}\frac{\partial}{\partial t}B_1(t,\tau)dT_t$$

对于 $z\in S^{+}$ 的情况，注意到图（1）有

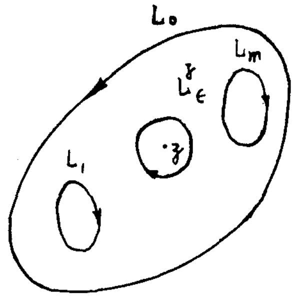
$$-\frac{1}{\pi}\iint_{S^-}\frac{\partial}{\partial t}\left[\frac{B_1(t,\tau)}{t-z}\right]dT_t=\frac{1}{2\pi i}\int_{L}\frac{B_1(\sigma,\tau)\overline{d\sigma}}{\sigma-z}-\lim_{\epsilon\to0}\int_{L_\epsilon}\frac{B_1(\sigma,\tau)\overline{d\sigma}}{\sigma-z}=$$

$$=\frac{1}{2\pi i}\int_{L}\frac{B_1(\sigma,\tau)\overline{\sigma'(s)^2}d\sigma}{\sigma-z}\text{，于是有}$$

$$\frac{\partial K_1}{\partial z}=\frac{1}{2\pi i}\int_{L}\frac{B_1(\sigma,\tau)\overline{\sigma'(s)^2}d\sigma}{\sigma-z}+\frac{1}{\pi}\iint_{S^-}\frac{1}{t-z}\frac{\partial}{\partial t}B_1(t,\tau)dT_t$$

$$\frac{\partial^2K_1}{\partial z^2}=\frac{1}{2\pi i}\int_{L}\frac{B_1(\sigma,\tau)\overline{\sigma'(s)^2}d\sigma}{(\sigma-z)^2}+\frac{1}{\pi}\iint_{S^-}\frac{1}{(t-z)^2}\frac{\partial}{\partial t}B_1(t,\tau)dT_t$$

对上式第二个积分性质是已知的[4]。现研究第一个积分，注意到围道L的弧长参数表示，$\sigma=\sigma(s)$，$0\leqslant s\leqslant l$，于是有



图（1）

$$\frac{1}{2\pi i}\int_L \frac{B_1(\sigma,\tau)\overline{\sigma'(s)^2}d\sigma}{(\sigma-z)^2} = -\frac{1}{2\pi i}\int_L B_1(\sigma,\tau)\overline{\sigma'(s)^2}\, d\left(\frac{1}{\sigma-z}\right)$$

$$= -\frac{1}{2\pi i}\int_L d\left(\frac{B_1(\sigma,\tau)\overline{\sigma'(s)^2}}{\sigma-z}\right)+\frac{1}{2\pi i}\int_L\frac{1}{\sigma-z}d[B_1(\sigma,\tau)\overline{\sigma'(s)^2}]$$

$$= -\frac{1}{2\pi i}\int_0^l \frac{d}{ds}\left\{\frac{B_1(\sigma(s),\tau)\overline{\sigma'(s)^2}}{\sigma(s)-z}\right\}ds +$$

$$+\frac{1}{2\pi i}\int_0^l\frac{1}{\sigma(s)-z}\frac{d}{ds}\{B_1(\sigma(s),\tau)\overline{\sigma'(s)^2}\}ds$$

$$= -\frac{1}{2\pi i}\left\{\frac{B_1(\sigma(s),\tau)\overline{\sigma'(s)^2}}{\sigma(s)-z}\right\}\Big|_0^l + \frac{1}{2\pi i}\int_L\frac{\overline{\sigma'(s)}\frac{d}{ds}[B_1(\sigma,\tau)\overline{\sigma'(s)^2}]d\sigma}{\sigma-z}$$

$$=\frac{1}{2\pi i}\int_L\frac{\overline{\sigma'(s)}\frac{d}{ds}[B(\sigma,\tau)\overline{\sigma'(s)^2}]d\sigma}{\sigma-z}$$

这里是由于L属于C_α^{N+1}的光滑封闭围道，故在上式中$\left\{\frac{B_1(\sigma(s),\tau)\overline{\sigma'(s)^2}}{\sigma(s)-z}\right\}\Big|_0^l$为零，故得

$$\frac{\partial^2 K_1}{\partial z^2}=\frac{1}{2\pi i}\int_L \frac{\overline{\sigma'(s)}\frac{d}{ds}[B(\sigma,\tau)\overline{\sigma'(s)^2}]d\sigma}{\sigma-z}+\frac{1}{\pi}\iint_{S^+}\frac{1}{(t-z)^2}\frac{\partial}{\partial t}B_1(t,\tau)dT_t。$$

下面用数学归纳法证明一般公式，设 $n-1$ 阶导数有

$$\frac{\partial^{n-1}K}{\partial z^{n-1}} = \frac{1}{2\pi i}\int_L \frac{\overline{\sigma'(s)}\overbrace{\frac{d}{ds}\{\overline{\sigma'(s)}\frac{d}{ds}\{\cdots\frac{d}{ds}}^{n-2\text{次}}[B_1(\sigma,\tau)\overline{\sigma'(s)^2}]\}\cdots\ d\sigma}{\sigma-z} +$$

$$+\frac{1}{2\pi i}\int_L\frac{\overline{\sigma'(s)}\overbrace{\frac{d}{ds}\{\overline{\sigma'(s)}\frac{d}{ds}\{\cdots\frac{d}{ds}}^{n-3\text{次}}\{\overline{\sigma'(s)}\ \{\frac{\partial}{\partial t}B_1(t,\sigma)\}_{t=\sigma}\overline{\sigma'(s)^2}\}\cdots\}d\sigma}{\sigma-z}$$

$$+\frac{1}{\pi}\iint_{S^+}\frac{1}{(t-z)^2}\frac{\partial^{n-2}}{\partial t^{n-2}}B_1(t,\tau)dT_t$$

同样应用上述方法将最后一项改写为

$$\frac{1}{\pi}\iint_{S^+}\frac{1}{(t-z)^2}\frac{\partial^{n-2}}{\partial t^{n-2}}B_1(t,\tau)dT_t = \frac{1}{2\pi i}\int_L\frac{\{\frac{\partial^{n-2}}{\partial t^{n-2}}B_1(t,\tau)\}_{t=\sigma}\overline{\sigma'(s)^2}d\sigma}{\sigma-z}+$$

$$+\frac{1}{\pi}\iint_{S^+}\frac{1}{t-z}\frac{\partial^{n-1}}{\partial t^{n-1}}B_1(t,\tau)dT_t$$

再对 z 求一次导数，经过计算最后得

$$\frac{\partial^nK_1}{\partial z^n}=\frac{1}{2\pi i}\int_L\frac{\overline{\sigma'(s)}\overbrace{\frac{d}{ds}\{\overline{\sigma'(s)}\frac{d}{ds}\{\overline{\sigma'(s)}\frac{d}{ds}\{\cdots\frac{d}{ds}}^{n-1\text{次}}[B_1(\sigma,\tau)\overline{\sigma'(s)^2}]\}\cdots\}d\sigma}{\sigma-z}$$

$$+\frac{1}{2\pi i}\int_L\frac{\overline{\sigma'(s)}\overbrace{\frac{d}{ds}\{\overline{\sigma'(s)}\frac{d}{ds}\{\overline{\sigma'(s)}\frac{d}{ds}\{\frac{d}{ds}}^{n-2\text{次}}\{\overline{\sigma'(s)}\{\frac{\partial}{\partial t}B_1(t,\tau)\}_{t=\sigma}\overline{\sigma'(s)^2}\}\cdots\}d\sigma}{\sigma-z}$$

$$+\frac{1}{2\pi i}\int_{L}\frac{\overline{\sigma'(s)}\frac{d}{ds}\left\{\left[\frac{\partial^{n-2}}{\partial t^{n-2}}B_1(t,\tau)\right]_{t=\sigma}\overline{\sigma'(s)^2}\right\}d\sigma}{\sigma-z}+$$

$$+\frac{1}{\pi}\int_{S^+}\frac{1}{(t-z)^2}\frac{\partial^{n-1}}{\partial t^{n-1}}B_1(t,\tau)dT_t \tag{16}$$

由[4]、[5]得知 $K_1(z,\tau)$ 本身以及直至 $n-1$ 阶导数均连续其 n 阶导数仅有对数型奇异性。当 $\tau\in L_j, j=1,2\cdots m$ 时

$$H_2(z,\tau)=\frac{(-1)^{n-1}}{(n-1)!}\ \frac{1}{\pi}\iint_{S^+}\frac{B(t)}{t-z}\ \overline{\frac{(\tau-t)^{n-1}}{(\tau-z_j)^n}\ln\left(1-\frac{\tau-z_j}{t-z_j}\right)}dT_t+\cdots$$

其奇异性讨论和（15）是类似的，对 $H_1(z,\tau)$ 的研究只需要利用性质(13)，即可证明相应的结论。

对 $W^-(z)$， $z\in S_j^-$，($j=1,2\cdots m$)，只要将（11）代入（6），经交换积分次序，可得上述同样的结果。

对 $W^-(z)$，$z\in S_0^-$，只要将（12）代入（9）经交换积分次序可得

$$W^-(z)=\frac{(-1)^p}{(P-1)!}\ \frac{1}{2\pi i}\int_{L^+}u(\tau)\frac{\prod_{k=1}^{m}{}'(\tau-z_k)^n}{m^{p+mn}}(\tau-z)^{p-1}\ln\left(1-\frac{\tau}{z}\right)d\tau+\cdots+$$

$$+\frac{1}{2\pi i}\int_{L_0^+}H_3(z,\tau)u(\tau)d\tau+\frac{1}{2\pi i}\int_{L_0}H_4(z,\overline{\tau)u(\tau)d\tau} \tag{17}$$

表达式（17）的第一部分，其 1 到 $P-2$ 阶导数的核函数是连续的，$P-1$ 阶导数出现对数型奇异性，P 阶导数出现柯西核，下面证明：$H_4(z,\tau)$ 其 1 到 $P-1$ 阶导数的核函数是连续的，P 阶导数出现对数型奇异性，$H_3(z,\tau)$ 其 1 到 P 阶导数的核函数均连续。

注意到 $H_4(z,\tau)=\frac{(-1)^{p-1}}{(P-1)!}\iint_{S_0^+}\Gamma_2(z,t,S_0^-)\ \overline{\frac{\prod_{k=1}^{m}{}'(\tau-z_k)^n(\tau-t)^{p-1}\ln\left(1-\frac{\tau}{t}\right)}{\tau^{p+mn}}}dT_t+\cdots,$

$\Gamma_1(z,t,S_0^-)=\Gamma_1^*\left(\frac{1}{z},\frac{1}{t},\overset{*}{S}{}_0^-\right)\frac{1}{|t|^4}$，$\Gamma_2(z,t,S_0^-)=\Gamma_2^*\left(\frac{1}{z},\frac{1}{t},\overset{*}{S}{}_0^-\right)\frac{1}{|t|^4}$ 以及对应于方程（7）的预解核的性质

$\Gamma_1^*(\zeta,\eta,\overset{*}{S}{}_0^-)\frac{1}{\pi}\overset{*}{B}(\eta)\Omega_2^*(\zeta,\eta,\overset{*}{S}{}_0^-)$，$\Gamma_2^*(\zeta,\eta,\overset{*}{S}{}_0^-)=\frac{1}{\pi}\overset{*}{B}(\eta)\Omega_1^*(\zeta,\eta,\overset{*}{S}{}_0^-)$

$\Omega_1^*(\zeta,\eta,\overset{*}{S}{}_0^-)=\frac{1}{\eta-\zeta}+O(\ln|\eta-\zeta|)$，$\Omega_2^*(\zeta,\eta,\overset{*}{S}{}_0^-)=O(\ln|\eta-\zeta|)$

于是我们有

$$H_4(z,\tau)=\frac{(-1)^{p-1}}{(P-1)!}\iint_{S_0^-}\frac{1}{\pi}\left\{-\bar{t}^2B(t)\left[\frac{1}{\frac{1}{t}-\frac{1}{z}}\right]\frac{1}{|t|^4}\overline{\frac{\prod_{k=1}^{m}{}'(\tau-z_k)^n}{\tau^{p+mn}}(\tau-t)^{p-1}\ln\left(1-\frac{\tau}{t}\right)}\right\}\cdot$$

$$\cdot dT_t+\cdots=\frac{(-1)^{p-1}}{(P-1)!\pi}\iint_{S_0^-}\frac{B(t)}{t(t-z)}\ \overline{\frac{\left[\prod_{k=1}^{m}{}'(\tau-z_k)^n(\tau-t)^{p-1}\ln\left(1-\frac{\tau}{t}\right)\right]}{\tau^{p+mn}}}dT_t+\cdots$$

同样讨论奇异性的阶时只要讨论第 1 项，为简单记

$$B_2(t,\tau)=\frac{(-1)^{p-1}}{(P-1)}\quad\frac{B(t)\prod\limits_{k=1}^{m}{}'(\tau-z_k)^n(\tau-t)^{p-1}\ln\left(1-\frac{\tau}{t}\right)}{t\tau^{p+1}},$$

同时记 $K_2(z,\tau)=\frac{z}{\pi}\iint\limits_{S_0^-}\frac{B_2(t,\tau)}{t-z}dT'$，下面证明某 1 到 $P-1$ 阶导数均连续，P 阶导数出现对数型奇异性

$$\frac{\partial K_2}{\partial z}=\frac{1}{\pi}\iint\limits_{S_0^-}\frac{B_2(t,\tau)}{t-z}\mathrm{d}T_t+\frac{z}{\pi}\iint\limits_{S_0^-}\frac{B_2(t,\tau)}{(t-z)^2}\mathrm{d}T_t=\frac{1}{\pi}\iint\limits_{S_0^-}\frac{B_2(t,\tau)}{t-z}\mathrm{d}T_t-$$

$$-\frac{z}{\pi}\iint\limits_{S_0^-}\frac{\partial}{\partial t}\left[\frac{B_2(t,\tau)}{t-z}\right]\mathrm{d}T_t+\frac{z}{\pi}\iint\limits_{S_0^-}\frac{1}{t-z}\frac{\partial}{\partial t}B_2(t,\tau)\mathrm{d}T_t$$

对于 $z\in S_0^-$ 的情况，注意到图（2）有

$$-\frac{z}{\pi}\iint\limits_{S_0^-}\frac{\partial}{\partial t}\left[\frac{B_2(t,\tau)}{t-z}\right]\mathrm{d}T_t=-\frac{z}{2\pi i}\int_{L_0}\frac{B_2(\sigma,\tau)\mathrm{d}\overline{\sigma}}{\sigma-z}+\lim_{R\to\infty}\frac{z}{2\pi i}\int_{L_R}\frac{B_2(\sigma,\tau)\mathrm{d}\overline{\sigma}}{\sigma-z}-$$

$$-\lim_{\epsilon\to0}\frac{z}{2\pi i}\int_{L_\epsilon}\frac{B_2(\sigma,\tau)\mathrm{d}\overline{\sigma}}{\sigma-z}=-\frac{z}{2\pi i}\int_{L}\frac{B_2(\sigma,\tau)\overline{\sigma'(s)^2}\mathrm{d}\sigma}{\sigma-z}$$

$$\frac{\partial K_2}{\partial z}=\frac{1}{\pi}\iint\limits_{S_0^-}\frac{B_2(t,\tau)}{t-z}\mathrm{d}T_t-\frac{z}{2\pi i}\int_{L_0}\frac{B_2(\sigma,\tau)\overline{\sigma'(s)^2}\mathrm{d}\sigma}{\sigma-z}+\frac{z}{\pi}\iint\limits_{S_0^-}\frac{1}{t-z}\frac{\partial}{\partial t}B_2(t,\tau)\mathrm{d}T_t$$

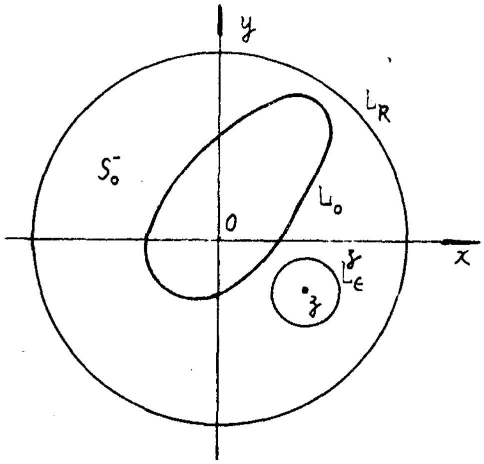
同理可得

$$\frac{\partial^2K_2}{\partial z^2}=-\frac{z}{2\pi i}\int_{L_0}\frac{\overline{\sigma'(s)}\frac{\mathrm{d}}{\mathrm{d}s}\left[B_2(\sigma,\tau)\overline{\sigma'(s)^2}\right]\mathrm{d}\sigma}{\sigma-\gamma}$$

$$-\frac{1}{\pi i}\int_{L_0}\frac{B_2(\sigma,\tau)\overline{\sigma'(s)^2}\mathrm{d}\sigma}{\sigma-z}$$

$$+\frac{2}{\pi}\iint\limits_{S_0^-}\frac{1}{\mathrm{t}-\mathrm{z}}\frac{\partial}{\partial\mathrm{t}}B_2(t_1)\mathrm{d}T_t$$

$$+\frac{z}{\pi}\iint\limits_{S_0^-}\frac{1}{(t-z)^2}\frac{\partial}{\partial t}B_2(t,\tau)\mathrm{d}T_t$$



图（2）

对一般情况，应用数学归纳法可证

$$\frac{\partial^pK_2}{\partial z^p}=-\frac{z}{2\pi i}\int_{L_0}\frac{\overline{\sigma'(s)}\frac{\mathrm{d}}{\mathrm{d}s}\overbrace{\left\{\sigma'(s)\frac{\mathrm{d}}{\mathrm{d}s}\left\{\cdots\frac{\mathrm{d}}{\mathrm{d}s}\left[B_2(\sigma,\tau)\overline{\sigma'(\tau)^2}\right]\right\}\cdots\right\}}^{p-1\text{ 次}}\mathrm{d}\sigma}{\sigma-z}$$

$$-\frac{z}{2\pi i}\int_{L_0}\frac{\overline{\sigma'(s)}\frac{\mathrm{d}}{\mathrm{d}s}\overbrace{\overline{\sigma'(s)}\frac{\mathrm{d}}{\mathrm{d}s}\left\{\cdots\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\partial}{\partial t}B_2(t,\tau)\right]_{t=\sigma}\right\}\cdots\Big\}}^{P-2\text{ 次}}\mathrm{d}\sigma}{\sigma-z}-$$

$$-\frac{1}{2\pi i}\int_{L_0}\frac{\overbrace{\overline{\sigma'(s)}\frac{d}{ds}\left\{\overline{\sigma'(s)}\frac{d}{ds}\left\{\cdots\frac{d}{ds}\left[B_2(\sigma,\tau)\overline{\sigma'(s)^2}\right]\right\}\cdots\right\}}^{P-2\text{次}}d\sigma}{\sigma-z}-\cdots+$$

$$+\frac{z}{\pi}\iint_{S_0^-}\frac{\frac{\partial^{p-1}}{\partial t^{p-1}}B_2(t,\tau)\,dT_t}{(t-z)^2} \tag{18}$$

由〔4〕、〔5〕命题得证，对 $H_3(z,\tau)$ 的证明是类似的。

利用表达式（14）、（17）以及 $W^-(z)$ 在 $z\in S_j^-$（$j=1,2\cdots m$）上相应的预解表达式，按边值条件（2）的需要求其对 z 的相应的各阶偏导数，再按沙霍斯基公式求边界值，依次将其代入边值条件（2）经整理最后导致下述奇异积分方程

$$\frac{a_n(t)}{2}(t-z_j)^{-n}u(t)+a_n(t)(t-z_j)^{-n}\frac{1}{2\pi i}\int_{L_j}\frac{u(\tau)d\tau}{\tau-t}+\frac{1}{\pi i}\int_{L_j}\frac{u(\tau)d\tau}{\tau-t}$$

$$\cdot\frac{1}{2}(\tau-z_j)^{-n}u(\tau)d\tau+\frac{1}{\pi i}\int_{L_j}\frac{R_n(t,t)}{\tau-t}\left[(\tau-z_j)^{-n}\frac{1}{2\pi i}\int_{L_j}\frac{u(\tau_1)d\tau_1}{\tau_1-\tau}\right]d\tau+\frac{b_p(t)}{2}(t-z_j)^{-n}$$

$$+\frac{b_p(t)}{2}(t-z_j)^{-n}\frac{\prod_{i=1}^{m}(t-z_i)^n}{t^{p+mn}}u(t)-\frac{b_p(t)}{2}(t-z_j)^{-n}\frac{\prod_{i=1}^{m}(t-z_i)^n}{t^{p+mn}}\frac{1}{\pi i}\int_{L_j}\frac{u(\tau)d\tau}{\tau-t}+$$

$$+\frac{1}{\pi i}\int_{L_j}\frac{S_p(t,\tau)}{\tau-t}(\tau-z_j)^{-n}\frac{\prod_{i=1}^{m}(\tau-z_i)^n}{\tau^{p+mn}}u(\tau)d\tau--\frac{1}{\pi i}\int_{L_j}\frac{S_p(t,\tau)}{\tau-t}\cdot$$

$$\cdot\left[\frac{\prod_{i=1}^{m}(\tau-z_i)^n}{\tau^{p+mn}}\frac{(\tau-z_j)^{-n}}{2\pi i}\int_{L_j}\frac{u(\tau_1)d\tau_1}{\tau_1-\tau}\right]d\tau+\text{核为弱奇性的项}=f(t)+\sum_{i=0}^{2n-1}c_iw_i(t)+$$

$$-\sum_{i=0}^{2p-1}a_i^jV_i^j(t),\quad t\in L_j,\quad j=1,2\cdots m, \tag{19}$$

$$\frac{a_n(t)}{2}u(t)+\frac{a_n(t)}{2\pi i}\int_{L_0}\frac{u(\tau)d\tau}{2\pi i}+\frac{1}{\pi i}\int_{L_0}\frac{R_n(t,\tau)}{\tau-t}\frac{1}{2}u(\tau)d\tau+$$

$$+\frac{1}{\pi i}\int_{L_0}\frac{R_n(t,t)}{\tau-t}\left[\frac{1}{2\pi i}\int_{L_0}\frac{u(\tau_1)d\tau_1}{\tau_1-\tau}\right]d\tau+\frac{b_p(t)}{2}\frac{\prod_{i=1}^{m}(t-z_i)^n}{t^{p+mn}}u(t)-$$

$$-\frac{b_p(t)}{2}\frac{\prod_{i=1}^{m}(t-z_i)^n}{t^{p+mn}}\frac{1}{\pi i}\int_{L_0}\frac{u(\tau)d\tau}{\tau-t}+\frac{1}{\pi i}\int_{L_0}\frac{S_p(t,\tau)}{\tau-t}\frac{\prod_{i=1}^{m}(\tau-z_i)^n}{\tau^{p+mn}}\frac{u(\tau)}{2}d\tau-$$

$$-\frac{1}{\pi i}\int_{L_0}\frac{S_p(t,\tau)}{\tau-t}\left[\frac{\prod_{i=1}^{m}(\tau-z_i)^n}{\tau^{p+mn}}\frac{1}{2\pi i}\int_{L_0}\frac{u(\tau_1)d\tau_1}{\tau_1-\tau}\right]d\tau+\text{核为弱奇性的项}=$$

$$=f(t)+\sum_{i=0}^{2n-1}c_iw_i(t),\quad t\in L_0 \tag{20}$$

此处 c_k+ic_{k+1} 为（10）中 $P_{n-1}(z)$ 的 k 次幂系数，$a_k^j+ia_{k+1}^j=d_{jk}$ 为（11）中 $n-1$ 次多项式的 k 次幂系数。

$$\left.\begin{aligned}
&w_i(t)=-\sum_{k=0}^{n}\left\{a_k(t)\frac{\partial^k W_i^+}{\partial t^k}+\frac{1}{i}\int_L\frac{R_k(t,\tau)}{\tau-t}\frac{\partial^k W_i^+}{\partial\tau^k}d\tau\right\},\quad t\in L,\ i=0,1,\cdots,2n-1;\\
&V_i^j(t)=\sum_{k=0}^{p}\left\{b_k(t)\frac{\partial^k \overset{(j)}{W_i^-}}{\partial t^k}+\frac{1}{\pi i}\int_{L_j}\frac{S_k(t,\tau)}{\tau-t}\frac{\partial^k \overset{(j)}{W_i^-}}{\partial\tau^k}d\tau\right\},\ t\in L_j,\ i=0,1,\cdots,2p-1,\\
&\qquad\qquad j=1,2,\cdots,m;\\
&k!\,\overset{(j)}{W}_{2k}(z)=z^k+\iint_{L_j^-}\Gamma_1(z,t,S_j^-)t^k dT_t+\iint_{S_j^-}\Gamma_2(z,t,S_j^-)\bar{t}^k dT_t,\quad j=1,2,\cdots,m,\\
&k!\,\overset{(j)}{W}_{2k+1}(z)=iz^k+\iint_{S_j^-}\Gamma_1(z,t,S_j^-)it^k dT_t+\iint_{S_j^-}\Gamma_2(z,t,S_j^-)\overline{it^k}dT_t,\ k=0,1,\cdots,p-1。
\end{aligned}\right\}\quad(21)$$

方程（19）、（20）经整理后可写为

$$A(t)u(t)+\frac{B(t)}{\pi i}\int_L\frac{u(\tau)d\tau}{\tau-t}+\int_L M_1(t,\tau)u(\tau)d\tau+\int_L M_2(t,\tau)u(\tau)d\tau=g(t),\quad t\in L\quad(22)$$

此处 $A(t)=\frac{1}{2}\left\{a_n(t)+b_p(t)\frac{\prod_{i=1}^{m}(t-z_i)^n}{t^{p+mn}}+R_n(t,t)-S_p(t,t)\frac{\prod_{i=1}^{m}(t-z_i)^n}{t^{p+mn}}\right\}$,

$$B(t)=\frac{1}{2}\left\{a_n(t)-b_p(t)\frac{\prod_{i=1}^{m}(t-z_i)^n}{t^{p+mn}}+R_n(t,t)+S_p(t,t)\frac{\prod_{i=1}^{m}(t-z_i)^n}{t^{p+mn}}\right\},$$

$$g(t)=\begin{cases}(t-z_j)^n\left[f(t)+\sum_{i=0}^{2n-1}c_iw_i(t)+\sum_{i=0}^{2n-1}a_i^jV_i^j(t)\right],& t\in L_j,\ j=1,2,\cdots,m\\ f(t)+\sum_{i=0}^{2n-1}c_iw_i(t),& t\in L_0\end{cases}$$

$M_i(t,\tau)$ $(i=1,2)$ 为确定的弗利特霍姆核。

由条件（4）我们得到

$$A(t)-B(t)=\frac{\prod_{i=1}^{m}(t-z_i)^n}{t^{p+mn}}[b_p(t)-S_p(t,t)]\neq 0,\ A(t)+B(t)=a_n(t)+R_n(t,t)\neq 0,\ t\in L\quad(23)$$

此时问题的指数定义为

$$\varkappa=\frac{1}{2\pi}\arg\left\{\frac{A(t)-B(t)}{A(t)+B(t)}\right\}_L=\frac{1}{2\pi}\arg\left\{\frac{\prod_{i=1}^{m}(t-z_i)^n}{t^{p+mn}}\frac{b_p(t)-S_p(t,t)}{a_n(t)+R_n(t,t)}\right\}_L=\varkappa_1-p-mn\quad(24)$$

此处 $\varkappa_1=\frac{1}{2\pi}\arg\left\{\frac{b_p(t)-S_p(t,t)}{a_n(t)+R_n(t,t)}\right\}_L$。

由（23）得知积分方程（22）是正则的，为了讨论可解性理论，引入对应的共轭齐次方程

$$A(t)\psi(t)-\frac{1}{\pi i}\int_L\frac{B(\tau)\Psi(\tau)d\tau}{\tau-t}+\int_L M_1(\tau,t)\psi(\tau)d\tau+\int_L\overline{M_2(\tau,t)\psi(\tau)d\tau}=0,\quad t\in L\quad(25)$$

设 k 为齐次方程（22）的解数，k' 为其共轭方程（25）的解数，由 Noether 理论有

$$k-k'=2\varkappa_1-2p-2mn\quad(26)$$

非齐次方程（22）可解的充要条件为

$$\mathrm{Re}\left\{\int_{L_0}\left[f(t)+\sum_{i=0}^{2n-1}c_iw_i(t)\right]\psi_j(t)dt+\sum_{q=1}^{m}\int_{L_q}(t-z_q)^n\left[f(t)+\sum_{i=0}^{2n-1}c_iw_i(t)+\right.\right.$$

$$+\sum_{i=0}^{2p-1} a_i^q V_i^q(t) \Big] \psi_j(t)\,dt \Big\} = 0, \quad j=1,2,\cdots k' \tag{27}$$

此处 $\psi_j(t)$，$j=1,2,\cdots k'$ 为方程（25）的线性完全无关解组，若记 $\mathrm{Re}\left\{\int_{L_0} f(t)\,\psi_j(t)\,dt + \sum_{q=1}^{m}\int_{L_q}(t-z_q)^n f(t)\,\psi_j(t)\,dt\right\} = -f_j$，　$\mathrm{Re}\left\{\int_{L_0} w_i(t)\psi_j(t)\,dt + \sum_{q=1}^{m}\int_{L_q} w_i(t)(t-z_q)^n\psi_j(t)\,dt\right\} = w_{ij}$，$\mathrm{Re}\left\{\int_{L_q}(t-z_q)^n V_i^q(t)\,\psi_j(t)\,dt\right\} = V_{ij}^q$。方程（27）可记为

$$\sum_{i=0}^{2n-1} c_i\, w_{ij} + \sum_{q=1}^{m}\sum_{i=1}^{2p-1} a_i^q V_{ij}^q = f_j, \quad j=1,2\cdots k' \tag{27}'$$

此方程组中 $\{c_i\}$、$\{a_i^q\}$ 为待定常数。

定理 1　齐次问题解的个数不小于 $2(m-1)p+2n(1-m)+2\varkappa_1$。

对 $n=p=1$ 的情况，边值问题有十分明确的几何及力学意义[4]，我们得到下述重要结果。

定理 2　在 $n=p=1$ 时，当 $\varkappa_1>0$ 时，齐次问题总有非零解。亦即对应黏合曲面允许无穷小变形，也就是非刚性的。刚性的情况仅可能出现在 $\varkappa_1<0$ 时

定理 3　非齐次问题可解的必充条件是方程（27）′的系数矩阵当增加常数项形成坛广矩阵其秩不增加。

参考文献

〔1〕Михайлов, Л. Г., Учен зап Тадж Ун-Та 10(1957).

〔2〕Михайлов, Л. Г., Изв. Выщ. Уч. Зав. Мамем, 5(1960), 99—109.

〔3〕闻国椿，应用数学与计算数学，1979. 6. 62—69.

〔4〕Векуа, И. Н., Обобщенные аналически функции, физматриз, 1959.

〔5〕Мусхелищвичи, Н. И, Сингулярные интегральные уравнение, ФМ, 1962.