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On The Spectral Decompositions of Spherical Matrix DistributionS and Some of Their Subclasses

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Abstract

In the viewpoint of spectral docomposition we study the spherical matrix distributions and obtain some new subclasses of Left-spherical distributions class. We also investigate rudely their properties and get some interesting examples.

Ⅰ. Introduction

This paper can be regarded as a continuation of our latest paper, Fang and Chen (1983). Hence, in the paper most notations remain the same as these of Fang and Chen (1983), but a few notations are altered to consist with other authors. $O(m)$ denotes the set of $m\times m$ orthogonal matrices and $V_{n\times p}=\{\Gamma(n\times p):\Gamma'\Gamma=I_p\}$ is the Stiefel manifold. If x and y have the same distribution, we write $x\overset{d}{=}y$. We call $x(n\times 1)$ having spherical distribution if $x\overset{d}{=}\Gamma x$ for each $\Gamma\in O(n)$, denoted by $x\sim S_n(\phi)$, where ϕ is x's c. f. (characteristic function). There are many different extensions of spherical distribution in the case of matrix. The following are several main classes involved in the paper.

$\mathcal{F}_L=\{X(n\times p):\Gamma X\overset{d}{=}X,\ \text{for each}\ \Gamma\in O(n)\}$,

$\mathcal{F}_S=\{X(n\times p):\Gamma XQ\overset{d}{=}X,\ \text{for each}\ \Gamma\in O(n),\ Q\in O(p)\}$,

$\mathcal{F}_2=\{X(n\times p):X=(x_1,\cdots,x_p)\overset{d}{=}(\Gamma_1x_1,\cdots,\Gamma_px_p),\ \text{fore ach}\ \Gamma_j\in O(n),j=1,\cdots,p\}$,

$\mathcal{F}_3=\{X(n\times p):\Gamma(\mathrm{Vec}X)\overset{d}{=}\mathrm{Vec}X,\ \text{for each}\ \Gamma\in O(np)\}$.

In the paper of Fang and Chen (1983), $\mathcal{F}_L$ appeared as $\mathcal{F}_1$. $\mathrm{Vec}X=(x_1',\cdots,x_p')'$ if $X=(x_1,\cdots,x_p)$. If for each $\Gamma\in O(m)$, $\Gamma Y\overset{d}{=}Y$, we cell also Y left-spherical. If Y' is left-spherical, Y is called right-spherical. We call Y spherical if it is both left-and right--spherical. So $\mathcal{F}_L$ is the set of $n\times p$ left-spherical random matrices. And $\mathcal{F}_S$ is the set of $n\times p$ spherical random matrices. We write $x\sim F$ to denote that x has distribution function F. Notation $\Upsilon_{n\times p}$ (appeared in Dawid (1977) as $\Upsilon_{n,p}$) denotes the unique uniform distribution on $V_{n\times p}$, especially, when $p=1$, $u^{(n)}\sim\Upsilon_{n\times 1}$ is the uniform distribution on the unit sphere in R^n, its c. f. (characteristic function) $\Omega_n(\cdot)$. Throughout this paper let $U\sim\Upsilon_{n\times p}$ that $U\in\mathcal{F}_S$ poihted out by Dawid (1977), and $A\geqslant 0$ denotes that A is an nonnegative definite matrix.

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We (1983) have obtained some relationships among $\mathcal{F}_L$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_S$, but, speaking roughlv, we dealt with them through coordinate decomposition. In this paper, we try to study them in the viewpoint of spectral decomposition so that more subclasses can easily be structured. May be these subclasses can help us to recongnise further the spherical matrix distributions system. In evidence, some interesting examples will be gotten by the subclasses. For example, we have known (Kelker (1970)) that if $\boldsymbol{x}(p\times 1)\sim S_p(\phi)$, then $\boldsymbol{x}\sim N(\boldsymbol{0}, a^2\boldsymbol{I}_p)$, provided $\boldsymbol{x}$'s some component (say x_i) $\sim N(0, a^2)$. Thus we can put a question: Does it hold yet in the case of matrix? That is, when $\boldsymbol{X}\in\mathcal{F}_S$ (or $\in\mathcal{F}_L$), $(n\geqslant p>1)$, if $\boldsymbol{X}$'s some row and column marginal distribution are normal, we are asked whether $\boldsymbol{X}$ is distributed according to matrix normal $N(\boldsymbol{0}, \boldsymbol{I}_n\otimes\Sigma)$. Perhaps one may conjecture the answer in the affirmative. In section 3, we shall discuss the problem in detail.

In section 2, we shall put the notion on spectral decomposition and give the spectral decompositions of $\mathcal{F}_L$, $\mathcal{F}_S$ and $\mathcal{F}_3$. Some new subclasses are obtained in section 3. In the last section, we define a new subclass of $\mathcal{F}_L$, and there are some interesting properties of those subclass. Unfortunately, we have not made it very clear the structure of the new subclass. Hence, on the occasion of the paper we put some problems we cann't solve to interested readers.

Ⅱ. Spectral Decomposition

Let's begin with a lemma that appeared in Dawid (1977), but, for its proof, he suggested reader to look up Dempster's book, however the system of his book is different from orthers, so we give an alternative proof here.

Lemma 1. If $\boldsymbol{X}(m\times k)$ is left-spherical (here, m and k are two arbitrary positive integers), then $\boldsymbol{X}$'s distribution is fully determined by that of $\boldsymbol{X}'\boldsymbol{X}$.

Proof. Suppose $\boldsymbol{Y}(m\times k)$ is also left-spherical and $\boldsymbol{Y}'\boldsymbol{Y}\overset{d}{=}\boldsymbol{X}'\boldsymbol{X}$, we want to show $\boldsymbol{X}\overset{d}{=}\boldsymbol{Y}$. Let $\gamma_{m\times m}$'s distribution and its characteristic function be F and $\Omega(\boldsymbol{T}'\boldsymbol{T})$, respectively, we have, $\boldsymbol{X}\sim$c. f. ϕ_X. $\boldsymbol{Y}\sim$c. f. ϕ_Y,

$$\phi_X(\boldsymbol{T}'\boldsymbol{T})=\phi_X(\boldsymbol{T}'\boldsymbol{T})\int_{O(m)}\mathrm{d}F=\int_{O(m)}\phi_X((\boldsymbol{UT})'(\boldsymbol{UT}))\mathrm{d}F(\boldsymbol{U})$$

$$=E\Big[\int_{O(m)}e^{i\mathrm{tr}\boldsymbol{T}'\boldsymbol{U}'\boldsymbol{X}}\mathrm{d}F(\boldsymbol{U})\Big]=E\Big[\int_{O(m)}e^{i\mathrm{tr}\boldsymbol{X}'\boldsymbol{UT}}\mathrm{d}F(\boldsymbol{U})\Big]$$

$$=E\Big[\int_{O(m)}e^{i\mathrm{tr}\boldsymbol{TX}'\boldsymbol{U}}\mathrm{d}F(\boldsymbol{U})\Big]=E[\Omega(\boldsymbol{TX}'\boldsymbol{XT}')]$$

$$=E[\Omega(\boldsymbol{TY}'\boldsymbol{YT}')]=\phi_Y(\boldsymbol{T}'\boldsymbol{T}),$$ for each $\boldsymbol{T}(m\times k)$, proving the lemma.

When $p=1$, Lemma 1 reduces essentically to Theorem 2 (Cambanis, Huang and Simons (1981)).

If $\boldsymbol{X}\in\mathcal{F}_L$, then $\boldsymbol{X}\overset{d}{=}\boldsymbol{UA}$, where $\boldsymbol{U}$ is defined in the section Ⅰ and is ind. (independent) of $\boldsymbol{A}$ $(\geqslant 0)$ (c. f. Dawid (1977)). Since $\boldsymbol{A}\geqslant 0$, $\boldsymbol{A}=\boldsymbol{V\Lambda V}'$, where

$V'V=I_p$, $\Lambda=\text{diag}(\lambda_1,\cdots,\lambda_p)$, $\lambda_1\geqslant\cdots\geqslant\lambda_p\geqslant 0$. We have, with $V=(r_1,\cdots,r_p)$,

$$X\overset{d}{=}UV\Lambda V'=\sum_{i=1}^{p}\lambda_i(Ur_ir_i')\equiv X_1+\cdots+X_p \tag{1}$$

say, and

$$X\overset{d}{=}U\Lambda V' \tag{2}$$

where U is ind of $\{\Lambda, V\}$ because in both of (1) and (2), X is left-spherical and $X'X\overset{d}{=}V'\Lambda^2V$ (cf. Lemma 1). We call equation (1) or (2) to be the spectral decomposition of X.

Property 1. $X_i'X_j=0$, $X_iX_j'=0$, $i\neq j$.

Property 2. $X_i'X_i/tr(X_i'X_i)$ is the orthogonally projective matrix on linear subspace $L(r_i)$, provided $\lambda_i<0$, $1\leqslant i\leqslant p$, where $L(B)$ denotes the linear subspace produced by B.

Property 3. $P_i=X_iX_i'/tr(X_i'X_i)$ is the orthogonally projective matrix on linear subspace $L(X_i)$, propided $\lambda_i>0$, $1\leqslant i\leqslant p$. And $P=P_1+\cdots+P_p$ is the orthogona lly projective matrix on $L(X)$, further, on $L(U)$ if $P(X'X>0)=1$.

Above properties's proofs are trival and omitted.

We should point out a fact that in general the spectral decomposition of X isn't unique and the next paragraph can be used as an illustration.

We consider a set $\mathcal{U}=\{(F_1,\cdots,F_p): F_i=Ur_ir_i', V=(r_1,\cdots,r_p)$ is ind of U, $V'V=I_p\}$. Clearly, for any $(F_1,\cdots,F_p)\in\mathcal{U}$, $U\overset{d}{=}F_1+\cdots+F_p=\sum_{i=1}^{p}Ur_ir_i'$ is the spectral decomposition of U. If particular, when $V\sim\gamma_{p\times p}$ we denote the $(F_1,\cdots,F_p)$ by $u_0=(E_1,\cdots,E_p)$ which is an especial element of $\mathcal{U}$.

Theorem 1. $X\in\mathfrak{F}_S$ iff $X\overset{d}{=}\lambda_1E_1+\cdots+\lambda_pE_p$, where $\lambda_1\geqslant\cdots\geqslant\lambda_p\geqslant 0$ is ind of $u_0=(E_1,\cdots,E_p)$.

Proof. If $X\in\mathfrak{F}_S$, then $X\overset{d}{=}U\Lambda U'$, $\Lambda=\text{diag}(\lambda_1,\cdots,\lambda_p)$, $\lambda_1\geqslant\cdots\geqslant\lambda_p\geqslant 0$, $V\sim\gamma_{p\times p}$, $U\sim\gamma_{n\times p}$ are all ind (cf. Dawid (1977)). (Note: if $V\sim\gamma_{n\times p}$, $V'=V^{-1}\overset{d}{=}V$). Hence $X\overset{d}{=}U\Lambda V'\overset{d}{=}UV\Lambda V'=\lambda_1E_1+\cdots+\lambda_pE_p$, where $E_i=Ur_ir_i'$, $i=1,\cdots,p$, $V=(r_1,\cdots,r_p)$ is ind. of U, and $u_0=(E_1,\cdots,E_p)$ is ind. of $(\lambda_1,\cdots,\lambda_p)$.

Conversely, if $(\lambda_1,\cdots,\lambda_p)$, is ind. of $u_0=(E_1,\cdots,E_p)$, then $X\overset{d}{=}\lambda_1E_1+\cdots+\lambda_pE_p=UV\Lambda V'\overset{d}{=}U\Lambda V'$ by the $(E_1,\cdots,E_p)$'s definition, where U, Λ, V are ind., $\Lambda=\text{diag}(\lambda_1,\cdots,\lambda_p)$ and $\Lambda\sim\gamma_{p\cdot p}$. That means $X\in\mathfrak{F}_S$, proving the theorem.

Before proceeding further it is maybe worth pointing out a fact. For each $X\in\mathfrak{F}_S$ there is an unique set of $\lambda_1\geqslant\cdots\geqslant\lambda_p\geqslant 0$ such that $X\overset{d}{=}\lambda_1E_1+\cdots+\lambda_pE_p$ and $(\lambda_1,\cdots,\lambda_p)$ ind. of $(E_1,\cdots,E_p)$. Hence we could regard $\{E_1,\cdots,E_p\}$ as a set of "base" in $\mathfrak{F}_S$ (even standard and orthogonal), and $(\lambda_1,\cdots,\lambda_p)$ as X's "coordinate" under the "base" $\{E_1,\cdots,E_p\}$. Basing on the observation, we can see that the spectral decomposition of X in $\mathfrak{F}_S$ shows the geometric significance of $X'X$'s latent roots $\{\lambda_1^2, \cdots, \lambda_p^2\}$ and describes how they determines fully the distribution of X.

Theorem 2. $X \in \mathfrak{F}_L$ iff $X \stackrel{d}{=} \lambda_1 F_1 + \cdots + \lambda_p F_p$, where $(F_1, \cdots, F_p) \in \mathcal{U}$ and $(\lambda_1 F_1 + \cdots + \lambda_p F_p | \lambda_1, \cdots, \lambda_p)$ is left-spherical given $(\lambda_1, \cdots, \lambda_p)$, $\lambda_1 \geqslant \cdots \geqslant \lambda_p \geqslant 0$.

Proof. If $X \in \mathfrak{F}_L$, X has the spectral decomposition (1) and let $F_i = U r_i r_i'$, $i = 1, \cdots, p$, then $(F_1, \cdots, F_p) \in \mathcal{U}$. Moreover $(\lambda_1 F_1 + \cdots + \lambda_p F_p | \lambda_1, \cdots, \lambda_p) = (U \Lambda V | \Lambda) \stackrel{d}{=} U(\Lambda V | \Lambda)$ is left-spherical since $U \in \mathfrak{F}_S$ and it is ind. of $\{\Lambda, V\}$.

Conversely, suppose $(F_1, \cdots, F_p) \in \mathcal{U}$ and $(\lambda_1 F_1 + \cdots + \lambda_p F_p | \Lambda)$ is left-spherical. Then, for each $\Gamma \in O(n)$, $\Gamma(\lambda_1 F_1 + \cdots + \lambda_p F_p | \Lambda) = (\Gamma(\lambda_1 F_1 + \cdots + \lambda_p F_p) | \Lambda) \stackrel{d}{=} (\lambda_1 F_1 + \cdots + \lambda_p F_p | \Lambda)$. Two random matrices' distributions are conditionally identical, so are their ones unconditionally. Thus $\Gamma(\lambda_1 F_1 + \cdots \lambda_p F_p) \stackrel{d}{=} \lambda_1 F_1 + \cdots + \lambda_p F_p$, for each $\Gamma \in O(n)$, proving the theorem.

The next theorem is a sequence of Theorem 1 and Fang & Chen's(1983)Theorem 7.

Theorem 3. $X \in \mathfrak{F}_3$ iff $X \stackrel{d}{=} \eta_1 E_1 + \cdots + \eta_p E_p$, where $u_0 = (E_1, \cdots, E_p)$ is ind of $(\eta_1, \cdots, \eta_p)$ and $\eta_i^2 = R^2 w_i / (w_1 + \cdots + w_p)$, $i = 1, \cdots, p$, $R \geqslant 0$ is ind. of $(w_1, \cdots, w_p)$ which are p latent roots of Wishart matrix $W \sim W_p(n, I_p)$.

We are not to give the spectral decomposition of X in $\mathfrak{F}_2$ since it is tedious. Having had Theorem 1, 2, 3, we can derive easily the spectral decomposition of $X'X$.

(i) If $X \in \mathfrak{F}_S$, then $X'X \stackrel{d}{=} \lambda_1^2 E_1' E_1 + \cdots + \lambda_p^2 E_p' E_p$, where $\lambda_1 \geqslant \cdots \geqslant \lambda_p \geqslant 0$ is ind. of $u_0 = (E_1, \cdots, E_p)$. Alternatively, $X'X \stackrel{d}{=} V' \Lambda V \stackrel{d}{=} V \Lambda V'$ with $V \sim \gamma_{p \times p}$, $\Lambda = \mathrm{diag}(\lambda_1^2, \cdots, \lambda_p^2)$, ind.

(ii) If $X \in \mathfrak{F}_L$, then $X'X \stackrel{d}{=} \lambda_1^2 F_1' F_1 + \cdots + \lambda_p^2 F_p' F_p$, where $(F_1, \cdots, F_p) \in \mathcal{U}$ and $(\lambda_1 F_1 + \cdots + \lambda_p F_p | \lambda_1, \lambda_2, \cdots, \lambda_p)$ is left-spherical. Alternatively, $X'X \stackrel{d}{=} V' \Lambda V$, with $\Lambda = \mathrm{diag}(\lambda_1^2, \cdots, \lambda_p^2)$ and $V'V = I_p$, but now it is unnecessary that $V \sim \gamma_{p \times p}$ and V is ind. of Λ. In fact we have.

Proposition 1. For orbitrary random matrices V, $V'V = I_p$, and $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_p)$, $\lambda_1 \geqslant \cdots \geqslant \lambda_p \geqslant 0$, there exists a unique $n \times p$ matrix X in $\mathfrak{F}_L$, $n \geqslant p$, such that $X'X \stackrel{d}{=} V' \Lambda V$.

Proof. Let $X = U \Lambda^{\frac{1}{2}} V$, where $U \sim \gamma_{n \times p}$ is ind. of $\{\Lambda, V\}$, $\Lambda^{\frac{1}{2}} = \mathrm{ding}(\lambda_1^{1/2}, \cdots, \lambda_p^{1/2})$. Then $X \in \mathfrak{F}_L$ and $X'X = V' \Lambda V$. If there exists another one (say) $Y \in \mathfrak{F}_L$, and $Y'Y \stackrel{d}{=} V' \Lambda V$, then $X'X \stackrel{d}{=} Y'Y$ implies $X \stackrel{d}{=} Y$ since the distribution of $X \in \mathfrak{F}_L$ is fully determined by that of $X'X$ (cf. Lemma 1). It follows.

(iii) If $X \in \mathfrak{F}_3$, then $X'X \stackrel{d}{=} \eta_1^2 E_1' E_1 + \cdots + \eta_p^2 E_p' E_p$, where $u_0 = (E_1, \cdots, E_p)$ is ind. of $(\eta_1, \cdots, \eta_p)$ that is defined by the Theorem 3.

We recall how it was complex deriving the distribution of the normalized characteristic vectors of $W \sim W_p(n, I_p)$. Now it is apparent because $W \stackrel{d}{=} X'X$, X

$\sim N\ (Q,\ I_n \otimes I_p)$, $X \in \mathcal{F}_S$ and $W \overset{d}{=} V' \Lambda V$ by (i), where $V \sim \gamma_{p\times p}$ is the normalized characteristic vectors matrix of W. In fact, writing $V=(v_{ij})$ and $J=\mathrm{diag}$ (sgn $v_{11},\cdots$, sgn v_{p1}) where sgn X deuotes the sign function and noting that $V\sim \gamma_{p\times p}$ and JV is the normalized characteristic vecters of W with the first column of JV being nonnegative, we immediately get the result that JV is distributed according to the conditional Haar invariant distribution which is Theorem 13.3. 3 in Anderson's book (1958). Further, for any $X \in \mathcal{F}_S$, the normalized characteristic vectors matrix of $X'X$ must be distributed according to $\gamma_{p\times p}$.

Ⅲ. Some Subclasses

For $X \in \mathcal{F}_L$, it has spectral decomposition (1), that is, $X \overset{d}{=} X_1 + \cdots + X_p$, where $X^i \in \mathcal{F}_L$ and $\mathrm{rk}(X_i)=\mathrm{rk}(\lambda_i)=\mathrm{sgn}\lambda_i$, $i=1,\cdots,\ p$. This fact arouses us to investigate the following class:

$$\mathcal{F}_4=\{X \in \mathcal{F}_L : \mathrm{rk}(X)=1,\ \text{a. e.}\ \}$$

In order to reserch $\mathcal{F}_4$ we consider some other sets:

$\mathcal{F}_4^{(1)}=\{X\ (n\times p)\ : X \overset{d}{=} Uyz'$, U is ind. of$\{y(p\times 1),\ z(p\times 1)\}.\}$,

$\mathcal{F}_4^{(2)}=\{X\ (n\times p)\ : X \overset{d}{=} yz'$, $y \in S_n(\phi)$, y is ind of $z(p\times 1).\}$,

$\mathcal{F}_4^{(3)}=\{X\ (n\times p)\ : X \overset{d}{=} u^{(n)}z'$, $u^{(n)}$ is ind of $z(p\times 1),\}$ and

$\mathcal{F}_4^{(4)}=\{X\ (n\times p)\ : X \overset{d}{=} U 1_n z'$, $z(p\times 1)$ ind. of $U\}$, where $1_n=(1,\cdots,\ 1)'$.

Theorem 4. $\mathcal{F}_4=\mathcal{F}_4^{(1)}=\mathcal{F}_4^{(2)}=\mathcal{F}_4^{(3)}=\mathcal{F}_4^{(4)}$.

Proof. $\mathcal{F}_4=\mathcal{F}_4^{(1)}$ is obvious by noting that $X \in \mathcal{F}_4$ implies $X \overset{d}{=} UA$, rk $(A)=1$, $A \geq 0$, a. e. and thus $X \overset{d}{=} Uyy'$, $A=yy'$, wherer $y(p\times 1)$ is ind. of U. Now we show $\mathcal{F}_4^{(3)}=\mathcal{F}_4^{(4)}$. $\mathcal{F}_4^{(3)} \supset \mathcal{F}_4^{(4)}$ is clea. If $X \in \mathcal{F}_4^{(3)}$, $X \overset{d}{=} u^{(n)}z' \overset{d}{=} U 1_n\ (z'/\sqrt{n})$ since $U 1_n \overset{d}{=} \sqrt{n}\, u^{(n)}$, where U can be chosen to be ind. of z, that is to say $X \in \mathcal{F}_4^{(4)}$. The rest of the proof leaves to reader.

Lemma 2. For $X \in \mathcal{F}_4$, if the distribution of $X\alpha$, where $\alpha \in R^p$ is a fixed constant vector, depends on α only through $\alpha'\alpha$, then $X \in \mathcal{F}_S$.

Rroof. As $\mathrm{rk}(X)=1$, a. e. we can write $X'X \overset{d}{=} yy'$, $y(p\times 1)$. Now, $\forall \alpha \in R^p$, $\Gamma \in O(p)$, we have $X\alpha \overset{d}{=} X\Gamma\alpha$ and thus $\alpha' X' X\alpha \overset{d}{=} \alpha' yy'\alpha \overset{d}{=} \alpha'\Gamma' X' X\Gamma\alpha \overset{d}{=} \alpha'\Gamma yy'\Gamma\alpha$ by the assumption. Take $\delta \sim \gamma_{1\times 1}$ (i. e. a random signal elemet) is ind. of y. As $\delta^2=\delta'\delta\equiv 1$, $\alpha' y\delta\cdot\delta y'\alpha \overset{d}{=} \alpha'\Gamma' y\delta\cdot\delta y'\Gamma\alpha$. Noting that $\delta\alpha' y$ and $\delta\alpha'\Gamma' y$ are all left-spherical (symmetric rardom variables), by Lemma 1, we get $\delta\cdot\alpha' y \overset{d}{=} \delta\alpha'\Gamma' y$ for each $\alpha \in R^p$ and $\Gamma \in O(p)$. Then $z \equiv \delta y \sim s_p\ (\phi)$ by noting that $Ee^{i\alpha' z}=Ee^{i(\Gamma'\alpha)' z}$ fhr each $\alpha \in R^p$ and $\Gamma \in O(p)$. Therefor, by $X'X \overset{d}{=} yy'=zz'$, $X'X$ is sphericl since $z\sim S_p(\phi)$, that is to say that X is right-spherical and X is spherical since X is left-spherical, proving the theorem.

By Lemma 2 we obtain the following theorem.

Theorem 5. $X \in \mathcal{F}_S \cap \mathcal{F}_4$ iff $X \in \mathcal{F}_4$ and the distribution of $X\alpha$, where $\alpha \in R_p$ fixed constant, depends upon α only through $\alpha'\alpha$. And we have

$\mathcal{F}_4 \cap \mathcal{F}_S = \{X(n\times p) : X \stackrel{d}{=} u^{(n)}z', \ u^{(n)}$ is ind. of $z \sim S_p(\phi).\}$.

Now let's go back to the spectral decomposition (2). If we have the Λ and V varied, we shall get many subclasses of $\mathcal{F}_L$. The first, we let $V \equiv I_p$, thus

$$\mathcal{F}_5 = \{X : X \stackrel{d}{=} U\Lambda, \ U \text{ is ind. of } \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_p), \ \lambda_i \geqslant 0, \ i=1, \cdots, p\}.$$

What information does $\mathcal{F}_5$ give us? Note that $X \in \mathcal{F}_5$ iff $X'X = \Lambda^2, X \in \mathcal{F}_L$. Therefore, the distribution of normalized characteristic vectors matrix of $X'X$ is a degenerate distribution.

Example 1. Take $\Lambda = \mathrm{diag}\ (R_1, \cdots, R_p)$, where $R_i \geqslant 0$, $i=1, \cdots, p$, and $(R_1^2, \cdots, R_p^2) \sim$ the generalized Rayleigh distribution, i. e. $(R_1^2, \cdots, R_p^2) \stackrel{d}{=} (w_{11}, \cdots, w_{pp})$, where $\{w_{ii}, \ 1 \leqslant i \leqslant p\}$ are the diagonal elements of $W \sim W_p\ (n, \Sigma)$. Then $X \stackrel{d}{=} U\Lambda \in \mathcal{F}_5$ and X's column marginal distributions are all normal, but that of X is not.

Example 2. Take $\Lambda = \mathrm{diag}\ (1/d_1, \cdots, 1/d_p)$ into $X \stackrel{d}{=} U\Lambda \in \mathcal{F}_5$, where $d_i > 0$, $i=1, \cdots, p$, $(d_1^2, \cdots, d_p^2) \sim D_p\left(\frac{1}{2}, \cdots, \frac{1}{2}\right)$. Now X's row marginal distribution is distributed according to 1-symmetric multivariate distibution, its c. f. belongs to $\Phi_p(1)$, studied by Cambanis, Keener and Simons (1981). In fact, if $u_{(1)}$ is the first row of U, $u_{(1)} \stackrel{d}{=} Ru^{(p)}$, where $u^{(p)}$ is ind. of $R^2 \sim \beta\left(\frac{p}{2}, \frac{n-p}{3}\right)$, (cf. Fang ang Chen (1983)), and $x_{(1)}$, the first row of X, $\stackrel{d}{=} R \cdot \Lambda \cdot u^{(p)}$, then $x_{(1)}$ has c. f. $\phi(|t_1| + \cdots + |t_p|) \in \Phi_p(1)$ since R is ind. of $\Lambda u^{(p)}$ and $\Lambda u^{(p)} \sim$ c. f. $\psi\ (|t_1| + \cdots + |t_p|)$ (cf. Cambanis, keener and Simons (1981), Th. 3.1.). But the c. f.s of columns of X have the form $\phi^*(t_1^2 + \cdots + t_p^2)$ which is dissributed according to 2-symmetric multivariate distribution.

A more general case is to take $\Lambda = R \cdot \mathrm{diag}(1/d, \cdots, 1/d_p)$ where $R \geqslant 0$ is ind. of $(d_1, \cdots, d_p)$ and $(d_1^2, \cdots, d_p^2)$ have the above meaning.

Put $\Lambda = RI_p$ into $\mathcal{F}_5$ or (1), we get

$$\mathcal{F}_5 = \{X : X \stackrel{d}{=} RU, \ U \text{ is ind. of } R \geqslant 0.\}.$$

Clearly, $\mathcal{F}_6 \subset \mathcal{F}_5$ and $\mathcal{F}_6 \subset \mathcal{F}_S$ by $U \in \mathcal{F}_S$. In the Introduction, we presented a question that given a random matrix $X \in \mathcal{F}_S$ with row and column being all normal, is it necessary that X is normal? The next example replies negatively to the question.

Example 3. In $\mathcal{F}_6$, we take $R^2 \sim \chi_n^2$, $X \stackrel{d}{=} RU$. Then X satisfies the followings (i) $X \in \mathcal{F}_S$. (ii) Each coulmn of $X = (x_1, \cdots, x_p)$ is normal $N(O, I_n)$. Let $U = (u_1, \cdots, u_p)$. Then $x_1 \stackrel{d}{=} Ru_1$, where $u_1 \stackrel{d}{=} u^{(n)}$ and $R_2 \sim \chi_n^2$ is ind. of u_1, implies $x_1 \sim N\ (O, I_n)$. (iii) Each row of $X = (x_{(1)}, \cdots, x_{(n)})'$ is normal. The reason is that $x_{(1)}$ is left-spherical. (i. e., $x_{(1)} \sim S_p\ (\phi)$) by $X \in \mathcal{F}_S$, and the components of $x_{(1)}$ are normal by (ii). (cf. Kelker (1970)). (iv) X is not normal. Otherwise X has independent components so that X has a pdf., (probability density function), but it is impossible when $p>1$. (cf. Fang and Chen (1983)).

Perhaps, Example 3 is not convincing because $\boldsymbol{X}$ has no pfd.. Therefore, we further consider the case of having a pdf.. That means that to find an $\boldsymbol{X}\in\mathcal{F}_S$, it has a paf. with row and column being all normal, but $\boldsymbol{X}$ is not normal.

Set $\boldsymbol{Y}\overset{d}{=}R\boldsymbol{U}^*$, where $\boldsymbol{U}^*=\gamma_{(n+p)\times p}$ is ind. of $R^2\sim\chi^2_{np}$. By using the above discussion, the each row and the each column $\boldsymbol{Y}$ has the normal distribution, and $\boldsymbol{Y}$ is not normal when $p>1$. Partition $\boldsymbol{Y}$ into two parts: $\boldsymbol{Y}_1(n\times p)$ and $\boldsymbol{Y}_2$ $(p\times p)$ and $\boldsymbol{Y}=(\boldsymbol{Y}_1', \boldsymbol{Y}_2')$. Then $\boldsymbol{Y}_1$ is just sought. Actually, it is easy to see that $\boldsymbol{Y}_1\in\mathcal{F}_S$, both row and column marginal distbutions are normal, and $\boldsymbol{Y}_1$ has a pdf. (cf. Fang and Chen (1983)). $\boldsymbol{Y}_1$, however, is not normal. If $\boldsymbol{Y}_1$ is normal, it must be $N(\boldsymbol{O}, \boldsymbol{I}_n\otimes\boldsymbol{I}_p)$, its c. f. is $\phi(\boldsymbol{T}_1'\boldsymbol{T}_1)=\exp\left(-\frac{1}{2}\operatorname{tr}\boldsymbol{T}_1'\boldsymbol{T}_1\right)$, where $\boldsymbol{T}_1$: $n\times p$. Then the c. f. of $\boldsymbol{Y}$ is also $\phi(\boldsymbol{T}'\boldsymbol{T})=\exp\left(-\frac{1}{2}\operatorname{tr}\boldsymbol{T}'\boldsymbol{T}\right)$ with $\boldsymbol{T}$: $(n+p)\times p$, but it is inpossible since $\boldsymbol{Y}$ is not normal. Thus $\boldsymbol{Y}_1$ couldn't be normal.

Before ending this section, we want to say a few words on the spectral decomposition again. For $\boldsymbol{X}\in\mathcal{F}_L$, $\boldsymbol{X}\overset{d}{=}\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}$. If we take both $\boldsymbol{\Lambda}$ and $\boldsymbol{V}$ to be constant, then such $\boldsymbol{X}$ is the projective transformation of $\boldsymbol{U}$. If we only fix the $\boldsymbol{\Lambda}$ to be constant, $\boldsymbol{X}'\boldsymbol{X}$'s latent roots are nonrandom and the normalized characteristic vectors matrix of $\boldsymbol{X}'\boldsymbol{X}$, however, can be random, but it is not necessary to be uniform distribufion $\gamma_{p\times p}$.

Ⅳ. Multidimensional Versions

The Lemma 2 reminds us of an intereresting notion introduced by Eaton (1981)——n-dimensional version of one-dimensional symmetric distributions: The distribution of a random vector $\boldsymbol{x}(n\times 1)$ is an n-dimensional version of the distribution of a symmetric random variable z if, for $\boldsymbol{t}\in R^n$, $\boldsymbol{t}'\boldsymbol{x}\overset{d}{=}c(\boldsymbol{t})z$ with the function $c(\boldsymbol{t})>0$ if $\boldsymbol{t}\neq 0$. Moreover Eaton (1981) has shown the following result.

Suppose z have an n-dimensional version and $\operatorname{Var}(z)<\infty$. Then every n-dimensional version of z is given by $\boldsymbol{A}\boldsymbol{x}_0$ where $\boldsymbol{x}_0\sim S_n(\phi)$ and $\boldsymbol{A}$ is an $n\times n$ nonsingular constant matrix.

Naturally, we may think what is an $n\times p$-matrix version of a vector $\boldsymbol{x}\in S_n(\phi)$. It seems to be reasonable to confine the $n\times p$-matrix versions within $\mathcal{F}_L$. Thus we can make up a definition as following.

Definition 1. $\boldsymbol{X}\in\mathcal{F}_L$ is called an $n\times p$-matrix version of $\boldsymbol{x}\sim S_n(\phi)$ if there exists a function $c(\cdot)$ on R^p to $[0, \infty)$ such that

(i) $c(a)>0$ if $a\neq 0$,

(ii) $Xa \overset{d}{=} c(a)x$, for each $a \in \mathrm{R}^p$.

Lemma 3. Suppose $X \sim S_n(\phi)$ with finite positive varance $a^2 = E(x'x)$. Then if $X \in \mathcal{F}_L$ is an $n \times p$-matrix version of x and $E(x_{(1)}x'_{(1)}) = \Sigma$, where $x^{(1)}$ is the first row of X, the function $c(a)$ corresponding to X is certainly $c(a)^2 = n \cdot a'\Sigma a/a^2$.

Proof. By Eaton (1981)'s technique, we have $Xa \overset{d}{=} c(a)x$, for each $a \in \mathrm{R}^p$ and $a'X'Xa \overset{d}{=} c(a)^2 x'x$, thus $E(a'X'Xa) = a'E(X'X)a = a'\left(\sum_{i=1}^n\right)E(x_{(i)}x'_{(i)})a = na'\Sigma a = c(a)^2 E(x'x) = a^2 c(a)^2$, i. e. $c(a)^2 = na'\Sigma a/a^2$, proving the lemma.

Our Lemma 2、3 have actually shown that if $\mathrm{rk}(X) = 1$ and $E(x_{(1)}x'_{(1)}) = I_p$, then X is an $n \times p$-matrix version of some $x \in S_n(\phi)$ iff $X \in \mathcal{F}_S$. But if we omit the assumption $\mathrm{rk}(X) = 1$, we don't know what will happen yet. Therefore we define a new set as following.

$\mathcal{F}_7 = \{X \in \mathcal{F}_L$: the distribution of Xa edpends on $a \in \mathrm{R}^p$ only through $a'a\}$.

We have not known whether "$\mathcal{F}_7 = \mathcal{F}_S$" holds or not. We think that it is probablely OK. At least, there are many usual properties in $\mathcal{F}_7$ that $\mathcal{F}_S$ has. It is clear that $\mathcal{F}_7 \supset \mathcal{F}_S$. The following are some simple facts on $\mathcal{F}_7$.

Theorem 5. $X \in \mathcal{F}_7$ iff $X \in \mathcal{F}_L$ and the first row $x_{(1)} \sim S_p(\phi)$ of X.

Proof. Suppose $X \in \mathcal{F}_7$. Then $X \in \mathcal{F}_L$ and $Xa \overset{d}{=} X\Gamma a$, for each $\Gamma \in O(p)$ and each $a \in \mathrm{R}^p$. Hence $a'\Gamma' x_{(1)} \overset{d}{=} a'x_{(1)}$, that is to say that $E[e^{ia'x(1)}] = E[e^{i(\Gamma a)'x(1)}]$ for each $\Gamma \in O(p)$ and each $a \in \mathrm{R}^p$. Let $x_{(1)}$'s c. f. be $\phi(t)$. Then $\phi(t) = \phi(\Gamma t)$, for each $\Gamma \in O(p)$ and each $a \in \mathrm{R}^p$, i. e., $x_{(1)} \sim S_p(\phi)$. Conversely, suppose $x_{(1)} \sim S_p(\phi)$ and $X \in \mathcal{F}_L$ with c. f. $\psi(T'T)$. Then $\psi(bb') = \phi(b'b)$ for each $b \in \mathrm{R}^p$ since $x_{(1)}$'s c. f. is also $\phi(b'b)$, $b \in \mathrm{R}^p$. Therefor, for any $a \in \mathrm{R}^p$, but fixed and any $t \in \mathrm{R}^p$, we have $E(e^{it'za}) = E(e^{itrat'z}) = \psi(t't \cdot aa') = \phi(t't \cdot a'a)$, and it shows that the distribution of Xa depends upon a only through $a'a$, completing the proof of theorem.

Corollary 1. $\mathcal{F}_3 = \mathcal{F}_7 \cap \mathcal{F}_2$.

Proof. clearly, $\mathcal{F}_3 \subset \mathcal{F}_7 \cap \mathcal{F}_2$. Now suppose $X \in \mathcal{F}_7 \cap \mathcal{F}_2$. Then $x_{(1)} \sim S_p(\phi)$ by Theorem 5, where $x_{(1)}$ is the first row of X, and $X \in \mathcal{F}_3$ since $X \in \mathcal{F}_2$ and $x_{(1)} \sim S_p(\phi)$. (cf. Fang and Chen (1983)'s Theorem 5). The assertion follows.

Proposition 2. Suppose $X = (x_{ij}) = (x_1, \cdots, x_p) = (x_{(1)}, \cdots, x_{(p)})' \in \mathcal{F}_7$ and have finite second moment. Then

(i) $\mathrm{Cov}(x_k, x_l) = \delta_{kl} \cdot a^2 I_n$, $k, l = 1, \cdots, p$, and

(ii) $\mathrm{Cov}(x_{(i)}, x_{(j)}) = \delta_{ij} \cdot a^2 I_p$, $i, j = 1, \cdots, n$,

where $\delta^{ij} = 0$ or 1 when $i \neq j$ or $i = j$.

Proof. We are only to show (ii) because the proof of (i) is similar to that of (ii). Noting $x_i \sim S_n(\phi)$ $i = 1, \cdots, p$, since $X \in \mathcal{F}_7$, we have $E(x_{i1}x_{j1}) = 0$, $i \neq j$, and $Xa \overset{d}{=} x_1$, for $a \in \mathrm{R}^p$ with $a'a = 1$. Thus $E(a'x_{(i)}x'_{(j)}a) = E(x_{i1}x_{j1}) = 0$, for $a \in \mathrm{R}^p$

with $a'a=1$, i. e., $a'\mathrm{Cov}(x_{(i)}, x_{(j)})\, a=0$ for each $a\in R^p$ and $i\neq j$. If we can point out that Cov $(x_{(i)}, x_{(j)})$ is a symmetric matrix, it must be zero matrix O, for $i\neq j$. In fact, $X\overset{d}{=}UA$, where U is ind. of A, $U=(u_1,\cdots, u_p)=(u_{(1)},\cdots, u_{(n)})'$. But $E(u_{(i)}u'_{(j)})=E(u_{(j)}u'_{(i)})$, $i\neq j$, and thus, given A, $E\ (A'u'_{(i)}u_{(j)}A)=E(A'u_{(j)}u'_{(i)}A)$, it implies $E(x_{(i)}, x'_{(j)})=E(x_{(j)}x'_{(i)})$ since A is ind. of U, that is to say that $\mathrm{Cov}(x_{(i)}, x_{(j)})'=\mathrm{Cov}(x_{(i)}, x_{(j)})$, proving (ii).

In the above statement, there is simple fact we haven't pointed out that is $x_1\overset{d}{=}\cdots\overset{d}{=}x_p$ if $X=(x_1, \cdots, x_p)\in\mathfrak{F}_7$. It is easy to see by $Xa\overset{d}{=}x_1$ for each $a\in R^p$ with $a'a=1$.

Proposition 3. Suppose $X\in\mathfrak{F}_7$ and $y(p\times 1)$ is ind. of X with $P(y'y=0)=0$. Then the distribution of $y'X'Xy/y'y$ is independet of that of y.

Proof. It is enough to show that the distribution of $a'X'aXa$, $a\in R^p$ and $a'a=1$, doesn't depend on a. But this is obvious by $X\in\mathfrak{F}_7$.

Proposition 4. The following statements are equivalent:

(i) $X\in\mathfrak{F}_7$ and $x_1\sim N(O, I_n)$, where $X=(x_1,\cdots, x_p)$.

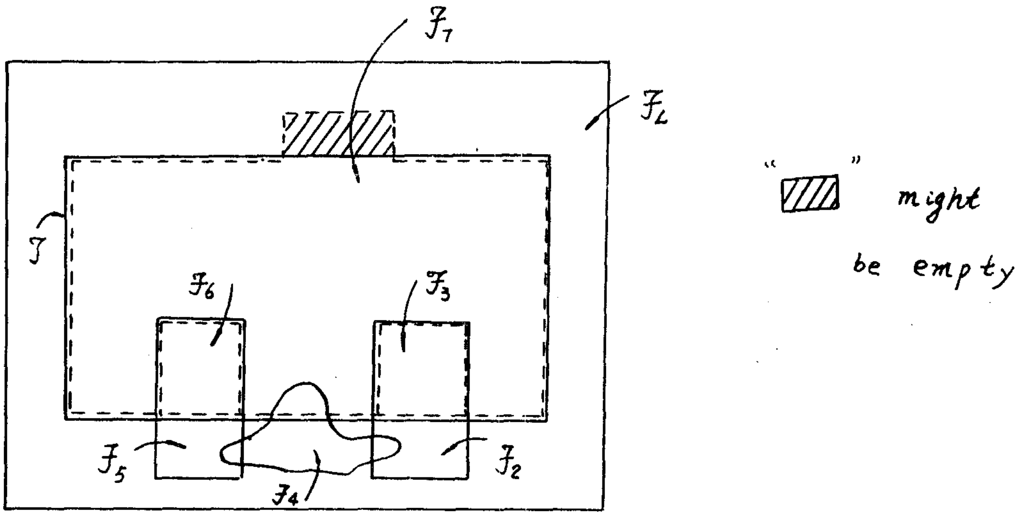
(ii) $X\in\mathfrak{F}_L$ and $a'X'Xa\sim\chi^2_n$, for each $a\in R^p$ with $a'a=1$.

Moreover, (i)or(ii) implies that the row and column marginal distrjbutions are all normal.

Proof. "(i) implies (ii)". As $X\in\mathfrak{F}_7$, we have $a'X'Xa\overset{d}{=}x'_1x_1\sim\chi^2_n$ by $x_1\sim N(O, I_n)$, for each $a\in R^p$ with $a'a=1$, implying (ii).

"(ii) implies (i)" Soppose $X\in\mathfrak{F}_L$ and $a'X'Xa\sim\chi^n_2$, for each $a\in R^p$ with $a'a=1$. It is easy to see that $Xa\sim N(O, I_n)$ because $Xa\in S_n(\phi)$ and $(Xa)'(Xa)\sim\chi^2_n$, for each $a\in R^p$ with $a'a=1$, and thus we can conclude that $X\in\mathfrak{F}_7$ since $Xa\sim N(O, a'aI_n)$ for each $a\in R^p$, proving that the proposition is true.

Finally, the following picture 1 gives a summary about the relationship among the classes of spherical matrix distributions.



Picture 1. Relationship among the classes of spherical matrix distbutions.

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